Published in
Proceedings of the Workshop on Ill-posed Variational Problems and Regularization Techniques, Trier (1998), L.N. in Economics and Mathematical Systems 477, Springer-Verlag (1999),pp. 24–35.

L^p REGULARIZATION OF THE NON-PARAMETRIC MINIMAL SURFACE PROBLEM

H. ATTOUCH AND T. CHAMPION

ABSTRACT. The non-parametric minimal surface problem is an ill-posed variational problem, even in its relaxed form. Indeed, the relaxed problem is an L^1 type problem, and it is not strictly convex so that it may have more than one solution. Following [2], we use the L^p regularization technique with $p \to 1$. Under fairly general assumptions, we show that the approximate solutions $(u_p)_p$ converge strongly in $W^{1,1}$ to a particular solution of the relaxed problem. Indeed, the so-selected solution is characterized as the unique solution of an auxiliary variational problem involving the integrand $t \mapsto |t|ln|t|$.

Keywords: L^p regularization, minimal surface problem, selection principle.

AMS subject classification: 35J70, 41A50, 41A65, 46E30, 49J30, 58E30

1. INTRODUCTION

In this paper, we are interested in a particular case of the minimal surface problem, namely the non-parametric minimal surface problem. This problem originated from the Plateau problem, which consists in finding the minimal surface (i.e. the surface of least area) bounded by a given closed curve in $\mathbb{R}^3$. In the so-called non-parametric minimal surface problem one has to determine a function u defined on a domain Ω of $\mathbb{R}^N$ whose graph is of least area among all the graphs of functions taking the same boundary values. This problem together with the minimal surface problem has been studied by many authors among which Radò, Douglas, De Giorgi, Federer, Fleming... For more precise and complete historical comments, we refer to [5].

The non-parametric minimal surface problem (P) is an ill-posed problem in its original statement, because it may have no solution taking the prescribed boundary values. Anyway, the boundary constraint can be relaxed so as to obtain a problem (P_1) which admits at least one solution, which can be viewed as a generalized solution of (P). But then (P_1) is still ill-posed because in general it has more than one solution. The aim of this paper is to provide with a way to regularize (P_1) in order to select a particular solution. This is done by defining a family of approximating well-posed problems $(P_p)_{p>1}$, which in our case will be the L^p-regularized problems of (P_1), with the property that if we denote u_p the solution

Date: January, 25 1999.

of (P_p), then $(u_p)_p$ is a minimizing sequence of (P_1) and converges to a particular solution of (P_1). In fact, the limit of $(u_p)_p$ is shown to solve an auxiliary well-posed minimization problem related to the regularization technique employed to define $(P_p)_p$.

2. Presentation of the problem and the L^p approximation

Throughout this paper, Ω is an open bounded connected subset of $\mathbb{R}^N$, with a Lipschitz continuous boundary that we shall denote $\partial\Omega$.

Let $g \in \mathrm{W}^{1,1}(\Omega) \cap \mathrm{L}^\infty(\Omega)$. Then the so-called non parametric minimal surface problem is:

$$(P) \qquad \operatorname{Inf}\left\{ \int_\Omega \sqrt{1+|\nabla u(x)|^2}dx \,:\, u \in g + \mathrm{W}_0^{1,1}(\Omega) \right\}$$

This problem is in general ill-posed in the sense that it doesn't always have a solution (see [3], chapter V example 2.1). In [3], Ekeland and Temam proposed a direct study of this problem via duality arguments, and defined a notion of generalized solution. In this paper, we follow De Giorgi, Giusti and Miranda (in [4]) and consider the following relaxed form of the non parametric minimal surface problem:

$$(P_1) \quad \operatorname{Inf}\left\{ J(u) = \int_\Omega \sqrt{1+|\nabla u(x)|^2}dx + \int_{\partial\Omega} |u-g|\, d\mathrm{H}_{N-1} \,:\, u \in \mathrm{W}^{1,1}(\Omega) \right\}$$

where H_{N-1} is the $N-1$-dimensional Hausdorff measure. We recall that each function u in $\mathrm{W}^{1,1}(\Omega)$ has a trace on $\partial\Omega$ (that we still denote by u) belonging to $\mathrm{L}^1(\partial\Omega)$.

Now the set $S(P_1)$ of optimal solutions of (P_1) is nonempty. This is not obvious because one would rather expect a minimizer of J to belong to $\mathrm{BV}(\Omega)$: when applying the direct method of the calculus of variations, one is led to imbed $\mathrm{W}^{1,1}(\Omega)$ into $\mathrm{BV}(\Omega)$ and to take the lsc-closure of J on $\mathrm{BV}(\Omega)$ (here we set $J(u) = +\infty$ for $u \in \mathrm{BV}(\Omega) \setminus \mathrm{W}^{1,1}(\Omega)$) so as to be able to extract a weakly convergent subsequence from a minimizing sequence of (P_1). The trick is that a solution obtained by this method (which is also a generalized solution in the sense of [3]) is analytic in Ω and belongs to $\mathrm{W}^{1,1}(\Omega)$, as shown in [5] (or [3]). The set $S(P_1)$ is also obviously convex, but may not be reduced to a singleton. For a counterexample, we refer to [5], example 15.12. Anyway, if g is continuous on $\partial\Omega$, then (P_1) has a unique optimal solution (see [5] and the reference therein for more details). In fact, as discussed in [3], if u_1 and u_2 are two optimal solutions of (P_1) then $\nabla u_1 = \nabla u_2$ because the member of J depending on the gradient is strictly convex. So $S(P_1)$ is a segment, and two optimal solutions of (P_1) differ only by a real constant. This also implies that if u_1 and u_2 are in $S(P_1)$, then $\int_{\partial\Omega} |u_1 - g|\, d\mathrm{H}_{N-1} = \int_{\partial\Omega} |u_2 - g|\, d\mathrm{H}_{N-1}$.

As explained in the introduction, the aim of this paper is to show that under suitable hypotheses, we can construct a family $(P_p)_{p>1}$ of well-posed variational problems such that if u_p is the solution of (P_p), then $(u_p)_p$ is a minimizing sequence of (P_1) which converges to a particular solution of (P_1).

One may first think of an elliptic type regularization (see [6]), and consider the family of problems:

$$(P_\varepsilon) \qquad \operatorname{Inf}\left\{ J_\varepsilon(u) = J(u) + \varepsilon \|u\|^2_{\mathrm{H}^1(\Omega)} \,:\, u \in \mathrm{H}^1(\Omega) \right\}$$

The trick is that for $\varepsilon > 0$, (P_ε) is elliptic on $\mathrm{H}^1(\Omega)$. Then if we denote by u_ε the unique solution of (P_ε), and suppose that $S(P_1) \cap \mathrm{H}^1(\Omega) \neq \emptyset$, it is easy to show that $(u_\varepsilon)_{\varepsilon>0}$ converges as $\varepsilon \to 0$ in $\mathrm{H}^1(\Omega)$ to the unique element $\tilde{u} \in S(P_1)$ which minimizes $\|.\|_{\mathrm{L}^2(\Omega)}$ over $S(P_1)$. Anyway, the condition $S(P_1) \cap \mathrm{H}^1(\Omega) \neq \emptyset$ does not seem natural since problem (P_1) is defined on $\mathrm{W}^{1,1}(\Omega)$. Moreover, when $S(P_1) \cap \mathrm{H}^1(\Omega) = \emptyset$ the convergence of $(u_\varepsilon)_{\varepsilon>0}$ to a particular solution of (P_1) is still an open problem. Noticing that, when $N = 2$ $\mathrm{W}^{1,1}(\Omega)$ is continuously imbedded in $\mathrm{L}^2(\Omega)$, an alternative approach to the elliptic regularization is to consider the following variational problem:

$$\mathrm{Inf}\left\{ J(u) + \varepsilon \|u\|^2_{\mathrm{L}^2(\Omega)} \ : \ u \in \mathrm{W}^{1,1}(\Omega) \right\}$$

Here, we will be interested in an other type of regularization, namely the L^p regularization, for which we can weaken the sufficient condition for the convergence of the net of minimizers. Following [2], we consider the family of well-posed approximating problems (P_p) related to (P_1):

$$(P_p) \quad \mathrm{Inf}\left\{ J_p(u) = \int_\Omega \left(\sqrt{1 + |\nabla u|^2} \right)^p dx + \int_{\partial\Omega} |u - g|^p \, d\mathrm{H}_{N-1} : u \in \mathrm{W}^{1,\mathrm{p}}(\Omega) \right\}$$

Now the functional J_p that appears in problem (P_p) is strictly convex, continuous and coercive on $\mathrm{W}^{1,\mathrm{p}}(\Omega)$ for $p > 1$, so (P_p) has a unique solution, that we will denote by u_p.We aim to show that $(u_p)_p$ is a minimizing sequence for (P_1), so that every cluster point of $(u_p)_p$ in $BV(\Omega)$ is an optimal solution of (P_1). Then, we will provide with a sufficient condition under which the whole net $(u_p)_p$ converges to a particular optimal solution. To this end, we need to define $\Phi : I\!R \to I\!R \cup \{+\infty\}$ by $\Phi(r) = \begin{cases} +\infty & \text{if } r < 0 \\ 0 & \text{if } r = 0 \\ r\ln(r) & \text{if } r > 0 \end{cases}$. Then Φ is convex on $I\!R^+$ and is everywhere not less than $-1/e$. For simplicity, we will use -1 as a lower bound for Φ in the calculations. Notice that $r \to \Phi(|r|)$ is *not* convex on $I\!R$.

The following theorem establishes that the family $(P_p)_p$ is a variational approximation of (P_1):

Theorem 2.1. *The sequence (u_p) is a minimizing sequence of (P_1) as $p \to 1$. Moreover,*

$$\lim_{p\to 1} J_p(u_p) = \lim_{p\to 1} J(u_p) = \mathit{Inf}(P_1)$$

Proof For every positive r, the function $p \mapsto r^p$ is convex on $I\!R^+$, with derivative $p \mapsto r^p \ln(r)$. Thus,

$$\forall r > 0 \quad \forall p > 1 \qquad r^p \geq r + (p-1) r \ln(r)$$

Now, let $u \in \mathrm{H}^1(\Omega)$ and $1 < p \leq 2$, then by the definition of u_p and the previous inequality

$$\begin{aligned} J_p(u) &\geq J_p(u_p) \\ &\geq J(u_p) + (p-1)\left[\int_\Omega \Phi\left(\sqrt{1+|\nabla u_p|^2}\right) dx + \int_{\partial\Omega} \Phi\left(|u_p - g|\right) d\mathrm{H}_{N-1}\right] \\ &\geq J(u_p) - (p-1)\left(\mathrm{L_N}(\Omega) + \mathrm{H_{N-1}}(\partial\Omega)\right) \\ &\geq \mathrm{Inf}(P_1) - (p-1)\left(\mathrm{L_N}(\Omega) + \mathrm{H_{N-1}}(\partial\Omega)\right) \end{aligned}$$

where $\mathrm{L_N}(\Omega)$ is the $N-1$-dimensional Lebesgue measure of Ω. Notice that $\mathrm{L_N}(\Omega)$ and $\mathrm{H}_{N-1}(\partial\Omega)$ are finite by hypothesis.
As $u \in \mathrm{H}^1(\Omega)$, we infer $\lim_{p\to 1} J_p(u) = J(u)$. By taking the limsup and the liminf as $p \to 1$ in the preceding inequalities and then taking the infimum over u in $\mathrm{H}^1(\Omega)$, which is dense in $\mathrm{W}^{1,1}(\Omega)$, we easily get our claim. □

Remark In fact, we can even show that J_p epiconverges to J on $\mathrm{W}^{1,1}(\Omega)$.

To be able to go further, we need to suppose some more regularity for the solutions of (P_1).

3. A sufficient condition for the convergence of $(u_p)_p$

As in [2], we introduce the following subspace of $\mathrm{W}^{1,1}(\Omega)$:

Definition 3.1. *We will denote* $\mathrm{W}^{1,1+}(\Omega) = \bigcup_{\mathrm{p}>1} \mathrm{W}^{1,\mathrm{p}}(\Omega)$

Remark The function $t \to 1/\ln(t)$ belongs to $\mathrm{W}^{1,1}(0,1/2) \setminus \mathrm{W}^{1,1+}(0,1/2)$, so in general $\mathrm{W}^{1,1+}(\Omega) \not\subset \mathrm{W}^{1,1}(\Omega)$.

With this in hand, we can state the main result of this paper, which is that if the elements of $S(P_1)$ are regular enough, then the whole net $(u_p)_p$ converges to a particular solution of (P_1). This result may be compared to the one we stated for the elliptic regularization in the preceding section, for which the natural hypothesis for the convergence of $(u_\varepsilon)_{\varepsilon>0}$ is $S(P_1) \cap \mathrm{H}^1(\Omega) \neq \emptyset$.

Theorem 3.2. *Assume that* $S(P_1) \cap \mathrm{W}^{1,1+}(\Omega) \neq \emptyset$.
Then the net $(u_p)_p$ *strongly converges in* $\mathrm{W}^{1,1}(\Omega)$ *to the unique solution* $\overline{u} \in S(P_1)$ *of the following auxilliary minimization problem:*

$$(P_1^+) \qquad \mathit{Inf}\left\{\int_{\partial\Omega} \Phi\left(|u-g|\right) dH_{N-1} \;:\; u \in S(P_1)\right\}$$

Remark Problem (P_1^+) can be considered as the selection principle linked to the L^p regularization of (P_1). Indeed, the solution of (P_1) selected as the limit of the approximate solutions $(u_p)_p$ is characterized as being the unique solution of (P_1^+).

Proof of theorem 3.2 As two solutions of (P_1) differ only by a real constant, $S(P_1) \subset \mathrm{W}^{1,1+}(\Omega)$. Let $q > 1$ be such that $S(P_1) \subset \mathrm{W}^{1,\mathrm{q}}(\Omega)$. Throughout the proof, we will assume $1 < p < q$.
Let $u \in S(P_1)$, then recalling the inequalities used in the proof of theorem 2.1, we get for all $1 < p < q$:

$$\begin{aligned} J_p(u) &\geq J_p(u_p) \\ &\geq J(u_p) + (p-1)\left[\int_\Omega \Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u_p - g|\right) d\mathrm{H}_{N-1}\right] \\ &\geq J(u) + (p-1)\left[\int_\Omega \Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u_p - g|\right) d\mathrm{H}_{N-1}\right] \end{aligned}$$

where, in the last inequality, we use that $u \in S(P_1)$. Let us rewrite the above inequality as

$$\begin{aligned} \int_\Omega \Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u_p - g|\right) d\mathrm{H}_{N-1} &\leq \frac{J_p(u) - J(u)}{p-1} \quad (1) \\ &\leq \frac{J_q(u) - J(u)}{q-1} \quad (2) \end{aligned}$$

the last inequality being a consequence of the convexity of $p \mapsto r^p$ for positive r.
Since the left hand side of (1) is uniformly bounded by the right hand side of (2), which is finite because $u \in S(P_1) \cap \mathrm{W}^{1,q}(\Omega)$, we may apply the Dunford-Pettis theorem to $(\nabla u_p)_p$. But as the family $(u_p)_p$ is also bounded in $\mathrm{W}^{1,1}(\Omega)$, we infer that $(u_p)_p$ is weakly $\mathrm{W}^{1,1}(\Omega)$-relatively compact. Let $(u_{p(k)})_{k\in \mathbb{N}}$,where $p(k) \to 1$, be a subsequence weakly converging in $\mathrm{W}^{1,1}(\Omega)$ to a function $u_1 \in \mathrm{W}^{1,1}(\Omega)$. As $(u_{p(k)})_{k\in\mathbb{N}}$ is a minimizing sequence of (P_1) and J is convex continuous on $\mathrm{W}^{1,1}(\Omega)$, we obtain $u_1 \in S(P_1)$.
Let us now show that u_1 is a solution of (P_1^+). Thanks to lemma 4.1, we know that $\left(\sqrt{1+|\nabla u_{p(k)}|^2}\right)_k$ weakly converges in $\mathrm{L}^1(\Omega)$ to $\sqrt{1+|\nabla u_1|^2}$ and $(|u_{p(k)} - g|)_k$ weakly converges in $\mathrm{L}^1(\partial\Omega)$ to $|u_1 - g|$.
As Φ is convex continuous on $\mathbb{R}^+$, we may pass to the liminf on the left hand side of (1) and obtain that for all $u \in S(P_1)$ and $p \in]1, q[$

$$\int_\Omega \Phi\left(\sqrt{1+|\nabla u_1|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u_1 - g|\right) d\mathrm{H}_{N-1} \leq \frac{J_p(u) - J(u)}{p-1}$$

Applying Lebesgue's monotone convergence theorem to the right hand side of this inequality, we get, for all u in $S(P_1)$

$$\begin{aligned} &\int_\Omega \Phi\left(\sqrt{1+|\nabla u_1|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u_1 - g|\right) d\mathrm{H}_{N-1} \\ &\qquad \leq \int_\Omega \Phi\left(\sqrt{1+|\nabla u|^2}\right)dx + \int_{\partial\Omega} \Phi\left(|u - g|\right) d\mathrm{H}_{N-1} \end{aligned}$$

As two solutions of $S(P_1)$ only differ by a constant, this results in

$$\forall u \in S(P_1) \qquad \int_{\partial\Omega} \Phi\left(|u_1 - g|\right) d\mathrm{H}_{N-1} \leq \int_{\partial\Omega} \Phi\left(|u - g|\right) d\mathrm{H}_{N-1}$$

So u_1 is a solution of (P_1^+). We claim that such a solution is unique. Indeed, let u and v be two optimal solutions of (P_1^+), then as $J(u) = J(v)$ and $\nabla u = \nabla v$, we easily get

$$\|u-g\|_{\mathrm{L}^1(\partial\Omega)}=\|v-g\|_{\mathrm{L}^1(\partial\Omega)}=\left\|\frac{u+v}{2}-g\right\|_{\mathrm{L}^1(\partial\Omega)} \tag{3}$$

As $S(P_1)$ and Φ are convex, we also obtain

$$\int_{\partial\Omega}\Phi\left(|u-g|\right)dx=\int_{\partial\Omega}\Phi\left(|v-g|\right)dx=\int_{\partial\Omega}\Phi\left(\left|\frac{u+v}{2}-g\right|\right)dx<\infty \tag{4}$$

Now, lemma 4.2 implies $u=v$ in $\mathrm{L}^1(\partial\Omega)$, so $u=v$ in $\mathrm{W}^{1,1}(\Omega)$. This proves our claim, and we obtain $u_1=\overline{u}$, where $\overline{u}$ is the unique solution of (P_1^+).
This implies that the whole net $(u_p)_p$ weakly converges to $\overline{u}$ in $\mathrm{W}^{1,1}(\Omega)$. It remains to show that it strongly converges to $\overline{u}$. To this end, we use inequality (1): we apply it with $u=\overline{u}$ and pass to the limsup as $p\to 1$ to get

$$\begin{aligned}\limsup_{p\to 1}&\left[\int_\Omega\Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx+\int_{\partial\Omega}\Phi\left(|u_p-g|\right)d\mathrm{H}_{N-1}\right]\\&\le\int_\Omega\Phi\left(\sqrt{1+|\nabla\overline{u}|^2}\right)dx+\int_{\partial\Omega}\Phi\left(|\overline{u}-g|\right)d\mathrm{H}_{N-1}\end{aligned}$$

Now, as $(u_p)_p$ weakly converges to $\overline{u}$ in $\mathrm{W}^{1,1}(\Omega)$, we apply lemmas 4.1 and 4.4 to obtain

$$\int_\Omega\Phi\left(\sqrt{1+|\nabla\overline{u}|^2}\right)dx\le\liminf_{p\to 1}\int_\Omega\Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx$$

and

$$\int_{\partial\Omega}\Phi\left(|\overline{u}-g|\right)d\mathrm{H}_{N-1}\le\liminf_{p\to 1}\int_{\partial\Omega}\Phi\left(|u_p-g|\right)d\mathrm{H}_{N-1}$$

so that

$$\lim_{p\to 1}\int_\Omega\Phi\left(\sqrt{1+|\nabla u_p|^2}\right)dx=\int_\Omega\Phi\left(\sqrt{1+|\nabla\overline{u}|^2}\right)dx$$

Then lemma 4.3 allows us to conclude that $(u_p)_p$ strongly converges to $\overline{u}$ in $\mathrm{W}^{1,1}(\Omega)$, thus finishing the proof. □

Remark Notice that problem (P_1^+) always makes sense since $S(P_1)\subset\mathrm{L}^\infty(\partial\Omega)$ (it is a consequence of $g\in\mathrm{L}^\infty(\Omega)$), and (P_1^+) is also always well-posed, even if $S(P_1)\cap\mathrm{W}^{1,1+}(\Omega)=\emptyset$. As shown in the previous proof, the unicity of the solution is a consequence of lemma 4.2. The existence of an optimal solution $\overline{u}$ of (P_1^+) can be easily shown by applying the direct method of the calculus of variations. Indeed, let $(u_n)_n$ be a minimizing sequence of (P_1^+), then by the Dunford-Pettis theorem we may extract a subsequence that weakly converges in $\mathrm{L}^1(\partial\Omega)$ to some $\overline{u}$, which then belongs to $S(P_1)$. Now the same trick as the one used in the proof of lemma 4.1 shows that $(|u_n-g|)_n$ also weakly converges in $\mathrm{L}^1(\partial\Omega)$ to $|\overline{u}-g|$, so that $\overline{u}$ is an optimal solution of (P_1^+) thanks to lemma 4.4. This naturally suggests the following question: is theorem 3.2 still valid without assuming $S(P_1)\cap\mathrm{W}^{1,1+}(\Omega)\neq\emptyset$?

4. Technical lemmas

We gather here some lemmas needed in the proof of theorem 3.2.

Lemma 4.1. *Suppose that* $(u_p)_{p>1}$ *weakly converges in* $\mathrm{W}^{1,1}(\Omega)$ *to* $u \in S(P_1)$. Then $\left(\sqrt{1+|\nabla u_p|^2}\right)_{p>1}$ *weakly converges in* $\mathrm{L}^1(\Omega)$ *to* $\sqrt{1+|\nabla u|^2}$ *and* $(|u_p-g|)_{p>1}$ *weakly converges in* $\mathrm{L}^1(\partial\Omega)$ *to* $|u-g|$.

Proof We first notice that $\lim_{p\to 1}\int_\Omega \sqrt{1+|\nabla u_p|^2}dx = \int_\Omega \sqrt{1+|\nabla u|^2}dx$ and $\lim_{p\to 1}\int_{\partial\Omega} |u_p-g|d\mathrm{H}_{N-1} = \int_{\partial\Omega}|u-g|d\mathrm{H}_{N-1}$. Indeed, this is an easy consequence of the weak lower semicontinuity of the functionals $\int_\Omega\sqrt{1+|\nabla .|^2}$ and $\int_{\partial\Omega}|.-g|\,d\mathrm{H}_{N-1}$ on the space $\mathrm{W}^{1,1}(\Omega)$ and of theorem 2.1 which asserts that $\lim_{p\to 1} J(u_p) = \mathrm{Inf}(P_1) = \int_\Omega\sqrt{1+|\nabla u|^2}dx + \int_{\partial\Omega}|u-g|d\mathrm{H}_{N-1}$.
Now, let A be a Borel subset of $\partial\Omega$, then

$$\int_A |u-g|d\mathrm{H}_{N-1} \quad\leq\quad \liminf_{p\to 1}\int_A |u_p-g|d\mathrm{H}_{N-1}$$

and

$$\begin{aligned}\limsup_{p\to 1}\int_A |u_p-g|d\mathrm{H}_{N-1} &= \limsup_{p\to 1}\Big[\int_{\partial\Omega}|u_p-g|d\mathrm{H}_{N-1} \\ &\qquad -\int_{\partial\Omega\setminus A}|u_p-g|d\mathrm{H}_{N-1}\Big]\\ &= \int_{\partial\Omega}|u-g|d\mathrm{H}_{N-1} - \liminf_{p\to 1}\int_{\partial\Omega\setminus A}|u_p-g|d\mathrm{H}_{N-1}\end{aligned}$$

so by the previous inequality, we get

$$\begin{aligned}\limsup_{p\to 1}\int_A |u_p-g|d\mathrm{H}_{N-1} &\leq \int_{\partial\Omega}|u-g|d\mathrm{H}_{N-1} - \int_{\partial\Omega\setminus A}|u-g|d\mathrm{H}_{N-1}\\ &= \int_A |u-g|d\mathrm{H}_{N-1}\end{aligned}$$

thus proving that $\lim_{p\to 1}\int_A |u_p-g|d\mathrm{H}_{N-1} = \int_A |u-g|d\mathrm{H}_{N-1}$. As this is true for every Borel subset A of $\partial\Omega$, the sequence $(|u_p-g|)_{p>1}$ weakly converges in $\mathrm{L}^1(\partial\Omega)$ to $|u-g|$.
We apply the same argument for $\left(\sqrt{1+|\nabla u_p|^2}\right)_{p>1}$. □

The three last lemmas can be stated in a more general setting, lemmas 4.3 and 4.4 being easy adaptations to the vectorial case of lemmas 6 and 7 in [7]. The following lemma implies the uniqueness of the solution (P_1^+).

Lemma 4.2. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a positive measure μ, and $\Psi : I\!\!R^+ \to I\!\!R$ a strictly convex function. Let u, v belong to $\mathrm{L}^1(\Omega)$ and satisfy (3') and (4').*

$$\|u\|_{\mathrm{L}^1(\Omega)} = \|v\|_{\mathrm{L}^1(\Omega)} = \left\| \frac{u+v}{2} \right\|_{\mathrm{L}^1(\Omega)} \qquad (3')$$

$$\int_\Omega \Psi\left(|u|\right) d\mu = \int_\Omega \Psi\left(|v|\right) d\mu = \int_\Omega \Psi\left(\left|\frac{u+v}{2}\right|\right) d\mu < \infty \qquad (4')$$

Then $u = v$ *in* $L^1(\Omega)$.

Proof It is easy to show that $(3')$ implies $\dfrac{|u(x)+v(x)|}{2} = \dfrac{|u(x)|+|v(x)|}{2}$ for μ-almost every $x \in \Omega$ (this is equivalent to say that $u(x)$ and $v(x)$ are of the same sign μ-almost everywhere on Ω). But then, as Ψ is strictly convex on $I\!\!R^+$, this and $(4')$ imply that $|u(x)| = |v(x)|$ for μ-almost every $x \in \Omega$. As a consequence, $u(x) = v(x)$ for μ-almost every $x \in \Omega$, which is our claim. □

Lemma 4.3. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a positive finite measure μ, and $C \subset I\!\!R^N$ be a closed convex set. Assume that $(h_n)_{n \in I\!\!N}$ is a sequence in $\mathrm{L}^1(\Omega)$ satisfying $h_n(\Omega) \subset C$ for all n. Let us assume that $(h_n)_n$ weakly converges to h in $\mathrm{L}^1(\Omega)$ and that there exists a continuous and strictly convex function $\Psi : C \to I\!\!R$ such that*

$$\limsup_{n \to +\infty} \int_\Omega \Psi \circ h_n d\mu \leq \int_\Omega \Psi \circ h d\mu \; < \; +\infty$$

Then (h_n) *strongly converges to h in* $\mathrm{L}^1(\Omega)$.

Lemma 4.4. *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space with a positive finite measure μ, and $C \subset I\!\!R^N$ be a closed convex set. Let also $\Psi : C \to I\!\!R$ be a continuous convex function.*

Assume that $(h_n)_{n \in I\!\!N}$ is a sequence in $\mathrm{L}^1(\Omega)$ satisfying $h_n(\Omega) \subset C$ for all n, and weakly converging to h in $\mathrm{L}^1(\Omega)$. Then

$$\int_\Omega \Psi \circ h d\mu \leq \liminf_{n \to +\infty} \int_\Omega \Psi \circ h_n d\mu$$

Remark In the proof of theorem 3.2, we apply these lemmas with $C = I\!\!R^+$ and $C = I\!\!R^N$.

References

[1] H. Attouch, *Variational convergence for functions and operators* (Pitman, London, 1984)

[2] H. Attouch and R. Cominetti, L^p Approximation of variational problems in L^1 and L^∞, *Nonlinear Analysis* 36 (1999) 373-399.

[3] I. Ekeland and R. Temam, *Analyse convexe et problèmes variationnels* (Dunod Gauthiers-Villars, Paris, 1974)

[4] E. DeGiorgi, Nouveaux résultats dans la théorie des hypersurfaces minima, *Conférences faites au Collège de France* juin 1970.

[5] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation* (Birkhauser, Boston, 1984)

[6] J.-L. Lions, *Quelques Méthodes de résolution des problèmes aux limites non linéaires* (Dunod et Gauthier-Villars, Paris, 1969)

[7] D. Landers and L. Rogge, Natural choice of L^1-Approximants, *Journal of Approximation Theory* 33 (1981) 268-280.

Laboratoire d'Analyse Convexe, Département de Mathématiques, case courrier 051, Université Montpellier II, Place Eugène Bataillon, 34095 Montpellier cedex, France. e-mails: attouch@math.univ-montp2.fr, champion@math.univ-montp2.fr