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PRINCIPLES OF COMPARISON WITH DISTANCE FUNCTIONS FOR ABSOLUTE MINIMIZERS.

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Abstract. We extend the principle of comparison with cones introduced by Crandall, Evans and Gariepy in [12] for the Minimizing Lipschitz Extension Problem to a wide class of supremal functionals. This gives a geometrical characterization of the absolute minimizers (optimal solutions whose minimality is local). Some application to the stability of absolute minimizers with respect to the Γ-convergence is given. A variation of the basic idea also allows to characterize the minimal Lipschitz extensions in length metric spaces.

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MSC 2000. 49K30, 65K10.

1. Introduction

In this paper we study the following problem

$$\min\left\{\operatorname*{ess.sup}_{x\in\Omega} H(x,Dv(x))\ :\ v\in g+W^{1,\infty}(\Omega)\cap\mathcal{C}_0(\Omega)\right\},\tag{1.1}$$

where Ω is a *connected* and *bounded* open subset of $\mathbb{R}^N$, H satisfies the natural assumptions to have the measurability of $H(.,Dv(.))$ for all $v\in W^{1,\infty}(\Omega)$, g is a map in $W^{1,\infty}(\Omega)\cap\mathcal{C}(\overline{\Omega})$.

This is a relevant class of variational problems associated with supremal functionals (see for example [15]). This type of functionals has received in the last few years a lot of attention because of many applications (see the bibliography of [1] for more details). A peculiarity of supremal functionals is the distinction between minimizers (defined as usually) and a class of particular minimizers, called absolute minimizers, defined as it follows:

Definition 1.1. *An absolute minimizer for (1.1) is a function $u\in W^{1,\infty}(\Omega)\cap\mathcal{C}(\overline{\Omega})$ such that $u=g$ on $\partial\Omega$ and for all open subset $V\subset\subset\Omega$ one has*

$$\operatorname*{ess.sup}_{x\in V} H(x,Du(x))\le\operatorname*{ess.sup}_{x\in V} H(x,Dv(x))$$

for all v in $W^{1,\infty}(V)\cap\mathcal{C}(\overline{V})$ such that $v=u$ on ∂V.

We recall that $V\subset\subset\Omega$ means that $\overline{V}\subset\Omega$, i.e. V is relatively compact in Ω. Notice that in the above definition, we restrict ourselves to the open subsets V which are relatively compact in Ω, which is the common definition for absolute minimizers (see [1]), but we do not assume that u is a minimizer of (1.1) which is in fact a consequence of this definition (see Lemma B.1 in the appendix). The existence of minimizers as well as absolute minimizers for problem (1.1) holds under mild assumptions, see [3, 4, 9]. Unlike usual minimizers of the problem (1.1), the absolute minimizer may be characterized as the unique solution of an associated PDE in the viscosity sense when the supremand H is smooth enough and satisfies some strict level convexity assumptions (we refer to [4, 10, 18]).

For the basic supremal functional

$$u \;\mapsto\; \|Du\|_{L^\infty(\Omega)} = \operatorname*{ess.sup}_{x\in\Omega} |Du(x)| \tag{1.2}$$

where the supremand is $H : (x,p) \mapsto |p|$, the absolute minimizers are also characterized by a geometric property introduced in [12] and called principle of comparison with cones. The principle of comparison with cones permitted some understanding toward the regularity of absolute minimizers of (1.2), also called ∞-harmonic functions. Using the principle of comparisons with cones it was recently proved in [20] that in 2 dimensions ∞-harmonic functions are C^1.

The aim of this paper is to provide a characterization of absolute minimizers of (1.1) through a generalized version of the Comparison with Cones that will be called Comparison with Distance Functions (see Definition 3.3 in §3). In fact, the "distance functions" we deal with are pseudo-distances whose definition involves the sublevel sets of H as well as the paths included in the open subsets $V \subset \Omega$ on which they are defined (see Definitions 2.3 and 2.5 in §2). Indeed, since the supremands H we consider depend on the variable x, the distance functions we use for the comparison must depend on the subset V on which that comparison holds (see Theorem 2.11 in §2).

The usual Comparison with Cones property for absolute minimizers of (1.2) as given in [1] is mainly based on the following fact. Let $V \subset \Omega \setminus \{x_0\}$ be open and consider the cone $u : x \mapsto a|x - x_0| + b$. If we denote by $Lip(u, \partial V)$ the Lipschitz constant of u on ∂V, then the maximal MacShane-Whitney extension u^+ of u from ∂V to $\overline{V}$, given by

$$u^+ \;:\; x \mapsto \inf\{u(y) + Lip(u, \partial V)|x - y| \;:\; y \in \partial V\},$$

is equal to u on $\overline{V}$. In the same spirit, the main tool of our Comparison with Distance Functions is to use the fact that, for any open subsets $V \subset \Omega$ and $U \subset\subset V$, one can still compare the distance function associated with H and V and its natural MacShane-Whitney extension on U (see Remark 2.12 in §2 and Proposition 3.1 in §3).

We show in §6 that the Comparison with Distance Functions may be adapted to the setting of length spaces (see Definition 6.2) and still characterizes absolute minimizers. We point out that our definition of Comparison with Distance Functions, when written in the special case of problem (1.2), suggests that when defining the Comparison with Cones from Above (see Definition 2.2 in [1]) one should consider cones $x \mapsto a|x - x_0| + b$ with a non-negative coefficient a (see Remark 6.3). This remark, in fact, ensures that our characterization for absolute minimizers holds even in the length space setting.

The importance of the relation between the supremal functional in (1.1) and the intrinsic distances associated with H was recently observed by many authors. In particular in [17], under the assumptions of homogeneity of H with respect to the gradient variable, the authors use the intrinsic distance function to characterize the relaxation of a supremal functional. In [16] for a C^2 supremand H which does not depend on x a comparison principle similar to the one we introduce here is investigated (see in particular section 4).

As final remark let us recall that we deal with an irregular function H then a-priori one cannot expect that the absolute minimizers are solutions of a PDE (see Example 3.2).

The paper is organized as follows: in section §2, we define the distances we use in the sequel and give a first upper-bound and lower-bound result for optimal solutions of problem (1.1) using these distances (see Theorem 2.11). Section §3 is devoted to the definition of the Comparison with Distance Functions (see Definition 3.3) and the main result of the paper, Theorem 3.5, that is the characterization of the absolute minimizers of problem (1.1) through the Comparison with Distance Functions. In section §4, we observe that when the supremand H satisfies some strict monotonicity in its second variable, it is possible to somewhat simplify the CDF, and thus to recover the classical notion of comparison with cones. We then apply the CDF characterization in §5 to the problem of stability of absolute minimizers with respect to the $\Gamma-$convergence of supremal functionals (see Theorems 5.1 and 5.3). We also show in section §6 that the Comparison with Distance Functions is easily adapted to the setting of length spaces and also allows to characterize the Absolutely Minimizing Lipschitz functions (see Theorem 6.4). In this paper, the concept of Finsler metrics plays an important role so we report some basic results in the Appendix A. The Appendix B is devoted to some technical results concerning the notions introduced in section §2.

2. Preliminary results

Throughout this work, we assume the following:

(A) $H \geq 0$, $H(\cdot, 0) = 0$ and $H(x, \cdot)$ is quasi-convex.

(B) H satisfies the following growth condition: $(x,p) \mapsto H(x,p)$ is uniformly (with respect to x) coercive in p, which means

$$\forall \lambda \geq 0 \quad \exists M \geq 0 \quad \forall (x,p) \in \Omega \times \mathbb{R}^N \qquad H(x,p) \leq \lambda \;\Rightarrow\; |p| \leq M.$$

(C) The map $(x,p) \mapsto H(x,p)$ is lower semi-continuous on $\Omega \times \mathbb{R}^N$.

We recall that $H(x, \cdot)$ quasi-convex means that any sublevel set $\{H(x, \cdot) \leq \lambda\}$ is convex. The hypotheses (A),(B) and (C) are rather standard and ensure the existence of absolute minimizers for problems of the type (1.1) (see [3, 4, 9]).

We now introduce the quasi-convex conjugate of H:

Definition 2.1. *For any $x \in \Omega$ and $\lambda \geq 0$, we define $L(x, \cdot, \lambda)$ on $\mathbb{R}^N$ by*

$$L(x, q, \lambda) := \sup \left\{ p \cdot q \,:\, p \in \mathbb{R}^N,\, H(x,p) \leq \lambda \right\}.$$

Notice that $L(x, q, \lambda) \geq 0$ for any $(x, q, \lambda) \in \Omega \times \mathbb{R}^N \times \mathbb{R}_+$, that it is positively 1-homogeneous and convex in q and that $L(\cdot, \cdot, \lambda)$ is measurable (by the upper semi-continuity w.r.t. x and the convexity w.r.t. to q, see appendix A). A function with such properties (for any fixed λ) is usually called a Finsler metric in Ω. We refer to the appendix for more about Finsler metrics and the associated distances.

Definition 2.2. *For any connected open subset $V \subset \Omega$ and any $x, y \in V$, we set*
$$\text{path}_V(x,y) := \left\{\xi \in W^{1,\infty}(]0,1[,V) \cap \mathcal{C}([0,1],V) \,:\, \xi(0) = x,\, \xi(1) = y\right\}.$$
In the case $V = \Omega$, we simply write $\text{path}_\Omega(x,y) = \text{path}(x,y)$.

We notice that since V is open and connected, $\text{path}_V(x,y)$ is nonempty for any $x, y \in V$. For further use, we recall the definition of the usual geodesic distance in V as given in [18] (see Definition 1.3 therein).

Definition 2.3. *For any connected open subset $V \subset \Omega$ and any $x, y \in V$, we define the metric distance $d^V(x,y)$ by*
$$d^V(x,y) := \inf\left\{\int_0^1 |\dot{\xi}(t)|dt \,:\, \xi \in \text{path}_V(x,y)\right\}.$$
and for any $x, y \in \overline{V}$ we set
$$d^V(x,y) := \inf\left\{\liminf_{n\to+\infty} d^V(x_n,y_n) \,:\, (x_n)_n, (y_n)_n \in V^{\mathbb{N}} \text{ and } x_n \to x, y_n \to y\right\}.$$
In the case $V = \Omega$, we simply write $d^\Omega(x,y) = d(x,y)$.

Remark 2.4. In the following, the usual distance between x and y in $\mathbb{R}^N$ shall be denoted by $|x-y|$, and the induced distance between a point x and a set A will be denoted by $dist(x,A)$.

By analogy with the definition above, we now introduce a family of pseudo-distances on the connected open subsets of Ω associated with H.

Definition 2.5. *For any connected open subset $V \subset \Omega$ and any $x, y \in V$ and any $\lambda \geq 0$, we set*
$$d^V_\lambda(x,y) := \inf\left\{\int_0^1 L(\xi(t), \dot{\xi}(t), \lambda)dt \,:\, \xi \in \text{path}_V(x,y)\right\},$$
and for any $x, y \in \overline{V}$ and any $\lambda \geq 0$, we set
$$d^V_\lambda(x,y) := \inf\left\{\liminf_{n\to+\infty} d^V_\lambda(x_n,y_n) \,:\, (x_n)_n, (y_n)_n \in V^{\mathbb{N}} \text{ and } x_n \to x, y_n \to y\right\}.$$
In the case $V = \Omega$, we simply write $d^\Omega_\lambda(x,y) = d_\lambda(x,y)$.

Remark 2.6. We point out here that since the boundary of V is not necessarily regular, one may have $d^V_\lambda(\tilde{x},y) = +\infty$ for some $\tilde{x} \in \partial V$ and $y \in V$: in this case, $d^V_\lambda(\tilde{x},y) = +\infty$ for any $y \in V$ due to the connectedness of V.

We notice that $\lambda \mapsto d^V_\lambda$ is non-decreasing, that d^V_λ is not a priori symmetric, but that it satisfies the triangular inequality
$$\forall x, y \in \overline{V}, \quad \forall z \in V \qquad d^V_\lambda(x,y) \leq d^V_\lambda(x,z) + d^V_\lambda(z,y). \tag{2.1}$$
The above inequality may be false for $z \in \partial V$ since we made no hypothesis on the regularity of the boundary of V. The same remarks hold for d^V.

We now establish a link between the usual notion of Lipschitz continuity on $V \subset \Omega$ with respect to that induced by the pseudo-distance d^V_λ.

Lemma 2.7. *Let V be a connected open subset of Ω. Assume that $u : \overline{V} \to \mathbb{R}$ satisfies*

$$\forall x, y \in V \qquad u(y) - u(x) \leq d_\lambda^V(x, y)$$

for some $\lambda \geq 0$. Then u belongs to $W^{1,\infty}(V) \cap \mathcal{C}(V)$.

If moreover u belongs to $\mathcal{C}(\partial V)$ and the inequality holds for $x, y \in \overline{V}$, then $u \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V})$.

Proof. It is sufficient to prove this in the case $V = \Omega$. Thanks to (B), there exists $M > 0$ such that $\{H(x, .) \leq \lambda\}$ is included in the Euclidean ball $B(0, M)$ for any x in Ω, so that $L(x, q, \lambda) \leq M|q|$ for any $x \in \Omega$ and $q \in \mathbb{R}^N$. As a consequence, $d_\lambda(x, y) \leq M|y - x|$ for any $x, y \in \overline{\Omega}$ such that the segment $]x, y[$ is included in Ω. Let $W \subset\subset \Omega$, if $\delta > 0$ denotes the distance from W to $\partial\Omega$, one then has

$$\forall x, y \in \overline{W} \text{ with } |y - x| \leq \delta \qquad u(y) - u(x) \leq M|x - y|$$

so that u is continuous on $\overline{W}$ and $\|Du\|_{L^\infty(W)} \leq M$, and since this holds for any $W \subset\subset \Omega$ we infer that u belongs to $W^{1,\infty}(\Omega) \cap \mathcal{C}(\Omega)$.

Now assume that $u \in \mathcal{C}(\partial\Omega)$, it remains to prove that $u(x_n) \to u(x)$ for any sequence $(x_n)_n$ of points in Ω converging to some $x \in \partial\Omega$. The following argument is borrowed from the proof of Theorem 1.8 in [18]. Let $(x_n)_n$ be such a sequence, and denote by y_n a projection of x_n on $\partial\Omega$ for the usual norm, then

$$\begin{aligned} |u(x_n) - u(x)| &\leq |u(x_n) - u(y_n)| + |u(y_n) - u(x)| \\ &\leq \max\{d_\lambda(x_n, y_n), d_\lambda(y_n, x_n)\} + |u(y_n) - u(x)| \\ &\leq M|x_n - y_n| + |u(y_n) - u(x)|. \end{aligned}$$

Since $x \in \partial\Omega$, one has $|x_n - y_n| \leq |x_n - x|$ so that $y_n \to x$, and since $u \in \mathcal{C}(\partial\Omega)$ we infer that $|u(y_n) - u(x)| \to 0$, which concludes the proof. □

Remark 2.8. The triangular inequality (2.1) yields that

$$\forall x, y \in V \quad d_\lambda^V(x_0, y) - d_\lambda^V(x_0, x) \leq d_\lambda^V(x, y)$$

for any connected open subset $V \subset \Omega$, any $\lambda \geq 0$ and $x_0 \in V$. Thus Lemma 2.7 also yields that the function $x \mapsto d_\lambda^V(x_0, x)$ is in $W^{1,\infty}(V) \cap \mathcal{C}(V)$. This also holds true when $x_0 \in \partial V$ if $x \mapsto d_\lambda^V(x_0, x)$ takes finite values in V. Notice that since no regularity hypothesis is made on $\partial\Omega$, even the distance $x \mapsto d_\Omega(x_0, x)$ may be discontinuous on $\partial\Omega$.

We now turn to the link between the essential supremum of $H(., Du(.))$ on V and the Lipschitzian character of the function u with respect to the pseudo-distance d_λ^V.

Proposition 2.9. *Let V be a connected open subset of Ω. Assume that $u \in W^{1,\infty}(V) \cap \mathcal{C}(V)$ is such that $H(\cdot, Du(\cdot)) \leq \lambda$ a.e. on V for some $\lambda \geq 0$. Then for any $x, y \in V$ one has $u(y) - u(x) \leq d_\lambda^V(x, y)$.*

Moreover, if $u \in \mathcal{C}(\overline{V})$ then $u(y) - u(x) \leq d_\lambda^V(x, y)$ holds for any $x, y \in \overline{V}$.

Proof. The first claim follows directly from Lemma B.3 of the appendix.

When $u \in \mathcal{C}(\overline{V})$, the final claim follows from the definition of d_λ^V by taking the liminf in $u(y_n) - u(x_n) \leq d_\lambda^V(x_n, y_n)$ with $x_n, y_n \in V$ for all n and $x_n \to x$, $y_n \to y$. □

The converse holds true:

Proposition 2.10. *Let V be a connected open subset of Ω. Assume that $u : V \to \mathbb{R}$ is such that for some $\lambda \geq 0$, $u(y)-u(x) \leq d_\lambda^V(x,y)$ for any $x, y \in V$. Then $H(\cdot, Du(\cdot)) \leq \lambda$ a.e. on V.*

Proof. It is sufficient to prove this in the case $V = \Omega$. We first infer from Lemma 2.7 that u belongs to $W^{1,\infty}(\Omega) \cap \mathcal{C}(\Omega)$, so that it is locally Lipschitz continuous on Ω. As a consequence, u is almost everywhere differentiable on Ω, and it is sufficient to show that $H(\cdot, Du(\cdot)) \leq \lambda$ for any $x \in \Omega$ at which u is differentiable.

Let thus $x \in \Omega$ be such that u is differentiable at x, then for any $q \in \mathbb{R}^N$ one has

$$\nabla u(x) \cdot q \;=\; \liminf_{h \to 0} \frac{u(x) - u(x-hq)}{h} \;\leq\; \liminf_{h\to 0} \frac{d_\lambda(x-hq, x)}{h}.$$

For $h > 0$ small enough, the function $\xi : [0,1] \to \mathbb{R}^N$ given by $\xi(t) := x - hq + thq$ belongs to $\text{path}(x - hq, x)$, so that

$$\frac{d_\lambda(x-hq,x)}{h} \;\leq\; \int_0^1 \frac{1}{h} L(x+(1-t)hq, hq, \lambda)dt \;=\; \int_0^1 L(x+(1-t)hq, q, \lambda)dt.$$

If for any $h > 0$ we set $f_h(t) := \sup\{L(x+(1-t)h'q, q, \lambda) \,:\, 0 < h' \leq h\}$, then the family $(f_h)_{h>0}$ converges to $\limsup_{h\to 0} L(x+(1-t)hq, q, \lambda)$ pointwise. By hypothesis (B) each f_h is dominated by $M|q|$ for some constant M, so that Lebesgue's dominated convergence theorem yields

$$\begin{aligned}\liminf_{h\to 0} \frac{d_\lambda(x-hq,x)}{h} \;\leq\; \lim_{h\to 0}\int_0^1 f_h(t)dt \;&=\; \int_0^1 \limsup_{h\to 0} L(x+(1-t)hq, q, \lambda)dt \\ &\leq\; \int_0^1 L(x,q,\lambda)dt = L(x,q,\lambda)\end{aligned}$$

the last inequality holding thanks to Lemma B.2. Therefore, $\nabla u(x) \cdot q \leq L(x, q, \lambda)$ for any $q \in \mathbb{R}^N$ and thus $\nabla u(x)$ belongs to the closed convex set $\{H(x, \cdot) \leq \lambda\}$. □

We now enlight the fundamental link between the pseudo-distances d_λ and the problem (1.1), which is a generalization of Lemma 1.6, Theorem 1.8 and the remark following that Theorem in [18].

Theorem 2.11. *Let V be a connected open subset of Ω, and consider the problem*

$$(P) \qquad \min\left\{F(v,V) := \operatorname*{ess.sup}_{x\in V} H(x, Dv(x)) \;:\; v \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V}), v = g \text{ on } \partial V\right\},$$

where g is a function in $W^{1,\infty}(V) \cap \mathcal{C}(\overline{V})$. Then the minimal value of this problem is

$$\mu := \min\left\{\lambda \,:\, g(y) - g(x) \leq d_\lambda^V(x,y) \text{ for any } x,y \in \partial V\right\}.$$

Moreover, the functions $S^-(g,V)$ and $S^+(g,V)$ given on $\overline{V}$ by

$$\forall x \in \overline{V} \quad S^-(g,V)(x) := \sup\{g(y) - d_\mu^V(x,y) \,:\, y \in \partial V\}$$

$$\forall x \in \overline{V} \quad S^+(g,V)(x) := \inf\{g(y) + d_\mu^V(y,x) \,:\, y \in \partial V\}$$

are optimal solutions of (P) and for any optimal solution u of (P) one has

$$\forall x \in \overline{V} \qquad S^-(g,V)(x) \leq u(x) \leq S^+(g,V)(x) \tag{2.2}$$

Notice that the minimal value μ for problem (P) is finite since $g \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V})$.

Remark 2.12. In the above statement one may take $V = \Omega$, so that this theorem provides a lower-bound and an upper-bound for the solutions of problem (1.1). The functions S^+ and S^- are obtained by analogy with the MacShane-Whitney operator (we refer to the introduction of [1] for more about this operator). We also attract the attention of the reader on the fact that in the expression of S^+ it appears $d_\mu^V(y,x)$ while in S^- there is $d_\mu^V(x,y)$: as d_μ^V is not symmetric, this is an important fact.

Proof. It is sufficient to prove this in the case $V = \Omega$. We first notice that the minimum μ need not *a priori* be attained, so that we shall at first set

$$\mu := \inf\{\lambda \,:\, g(y) - g(x) \leq d_\lambda(x,y) \;\; for\ any\ x, y \in \partial\Omega\}$$

as well as

$$\forall x \in \overline{\Omega} \quad S^-(x) := \sup\{g(y) - d_\lambda(x,y) \,:\, \lambda > \mu,\, y \in \partial\Omega\},$$
$$\forall x \in \overline{\Omega} \quad S^+(x) := \inf\{g(y) + d_\lambda(y,x) \,:\, \lambda > \mu,\, y \in \partial\Omega\}.$$

We first claim that $S^-(x) = g(x)$ for any $x \in \partial\Omega$. Indeed, taking $y = x$ in the definition of S^- yields $S^-(x) \geq g(x)$, while by definition of μ one has $g(y) - d_\lambda(x,y) \leq g(x)$ for any $\lambda > \mu$ and $y \in \partial\Omega$, so that $S^-(x) \leq g(x)$, which in turns proves the claim. The same holds for S^+.

We now prove that for any $\sigma > \mu$ and $x, y \in \overline{\Omega}$ one has

$$S^-(y) - S^-(x) \leq d_\sigma(x,y).$$

Indeed, take $\sigma > \mu$, $x \in \overline{\Omega}$ and $y \in \Omega$. We notice that since $\lambda \mapsto d_\lambda$ is non decreasing, the supremum in the definition of S^- can be taken for $\lambda \in]\mu,\sigma]$ instead of $\lambda > \mu$, so that

$$\begin{aligned} S^-(y) - S^-(x) &= \sup_{z\in\partial\Omega,\sigma\geq\lambda>\mu}\; \inf_{z'\in\partial\Omega,\sigma\geq\lambda'>\mu} \{g(z) - d_\lambda(y,z) - g(z') + d_{\lambda'}(x,z')\} \\ &\leq \sup_{z\in\partial\Omega,\sigma\geq\lambda>\mu} \{g(z) - d_\lambda(y,z) - g(z) + d_\lambda(x,z)\} \\ &\leq \sup_{\sigma\geq\lambda>\mu} \{d_\lambda(x,y)\} \;=\; d_\sigma(x,y) \end{aligned}$$

where we have applied inequality (2.1) which holds since $y \in \Omega$.

When $y \in \partial\Omega$, we notice that $S^-(y) = g(y)$ and $S^-(x) \geq g(y) - d_\sigma(x,y)$ so that the claim also holds. The corresponding estimate also holds for S^+.

Since g is continuous on $\partial\Omega$, we infer from the two preceding claims and Lemma 2.7 that S^- and S^+ belong to $g + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$. Moreover, it follows from Proposition 2.10 that $F(S^-,\Omega) \leq \sigma$ for any $\sigma > \mu$, so that $F(S^-,\Omega) \leq \mu$. Applying now Proposition 2.9 yields that

$$S^-(y) - S^-(x) \leq d_\mu(x,y)$$

for any $x, y \in \overline{\Omega}$, and since $S^- = g$ on $\partial\Omega$ this implies that the infimum in the definition of μ is attained.

Now the same arguments as above yield that $S^-(g,\Omega)$ and $S^+(g,\Omega)$ are admissible for problem (P) and that $F(S^-(g,\Omega),\Omega) \leq \mu$, so that the minimal value $\inf(P)$ for problem (P) is lower than μ. We claim that $\inf(P) = \mu$: by contradiction, assume that an admissible function $u \in g + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$ is such that $F(u,\Omega) \leq \lambda$ for some $\lambda < \mu$. Then Proposition 2.9 yields that $u(y) - u(x) \leq d_\lambda(x,y)$ for any $x, y \in \overline{\Omega}$, and since $u = g$ on $\partial\Omega$ this contradicts the definition of μ. As a consequence, $\inf(P) = \mu$ and $S^-(g,\Omega)$ and $S^+(g,\Omega)$ are optimal solutions of (P).

Finally, if u is an optimal solution of (P), one has $H(\cdot, Du(\cdot)) \leq \mu$ a.e. on Ω so that by Proposition 2.9 one gets $u(y) - u(x) \leq d_\mu(x,y)$ for any $x, y \in \overline{\Omega}$. If $x \in \overline{\Omega}$, this yields $g(y) - d_\mu(x,y) \leq u(x)$ for any $y \in \partial\Omega$ so we infer $S^-(g,\Omega)(x) \leq u(x)$. The same argument yields the estimate $u \leq S^+(g,\Omega)$ on $\overline{\Omega}$, which concludes the proof of (2.2). □

3. The comparison with distance functions

In the following Proposition, we relate the distance functions $d_\lambda^V(x_0,\cdot) + \alpha$ associated with H with the upper and lower solutions given in Theorem 2.11.

Proposition 3.1. *Let V be a connected open subset of Ω, and U be a connected open subset of V such that $U \subset\subset V$ and $x_0 \in \overline{V} \setminus U$. Then for any $\lambda \geq 0$ and $\alpha \in \mathbb{R}$, one has*

$$d_\lambda^V(x_0,\cdot) + \alpha \geq S^+(d_\lambda^V(x_0,\cdot) + \alpha, U) \ \text{ on } \overline{U},$$

and

$$-d_\lambda^V(\cdot, x_0) + \alpha \leq S^-(-d_\lambda^V(\cdot,x_0) + \alpha, U) \ \text{ on } \overline{U}.$$

Proof. It is sufficient to prove the first inequality in the case $V = \Omega$. We notice that by the connectedness of Ω either $d_\lambda(x_0,.)$ is identically $+\infty$ on $\overline{\Omega}$ (see Remark 2.6), or it is Lipschitz continuous on $\overline{U}$ (see Remark 2.8). In the first case there is nothing to prove, so we turn to the second case. We infer from Theorem 2.11 that for all x in $\overline{U}$:

$$S^+(d_\lambda(x_0,.) + \alpha, U)(x) := \inf\{\alpha + d_\lambda(x_0,y) + d_\mu^U(y,x) \ : \ y \in \partial U\}$$

where $\mu = \min\left\{\sigma \ : \ d_\lambda(x_0,y) - d_\lambda(x_0,x) \leq d_\sigma^U(x,y) \ for\ any\ x,y \in \partial U\right\}$. We observe that $\mu \leq \lambda$ because by (2.1) we have that for all $x, y \in \partial U$, $d_\lambda(x_0,y) - d_\lambda(x_0,x) \leq d_\lambda(x,y)$, and since $\text{path}_U(x,y) \subset \text{path}(x,y)$ one has $d_\lambda(x,y) \leq d_\lambda^U(x,y)$. Then

$$S^+(d_\lambda(x_0,.) + \alpha, U)(x) \leq \inf\{\alpha + d_\lambda(x_0,y) + d_\lambda^U(y,x) \ : \ y \in \partial U\} \tag{3.1}$$

for any $x \in U$. Now fix $x \in U$ and $\delta > 0$, and consider a path $\xi \in path(x_0,x)$ for which

$$d_\lambda(x_0,x) \geq \int_0^1 L(\xi(t),\dot{\xi}(t),\lambda)dt - \delta.$$

Then there exists $t \in [0,1[$ such that $\xi(t) \in \partial U$ and $\xi(s) \in U$ for any $s > t$, so that

$$\begin{aligned} d_\lambda(x_0,x) \ &\geq \ \int_0^t + \int_t^{t+\frac{1}{n}} + \int_{t+\frac{1}{n}}^1 L(\xi(t),\dot{\xi}(t),\lambda)dt - \delta \\ &\geq \ d_\lambda(x_0,\xi(t)) + 0 + d_\lambda^U(\xi(t+\frac{1}{n}),x) - \delta \end{aligned}$$

for any $n \geq 1$ for which $t + \frac{1}{n} < 1$. Taking the liminf as n go to $+\infty$ yields $d_\lambda(x_0,x) \geq d_\lambda(x_0,\xi(t)) + d_\lambda^U(\xi(t),x) - \delta$. Taking then $y = \xi(t)$ as a test in (3.1) yields

$$S^+(d_\lambda(x_0,.) + \alpha, U)(x) \leq \alpha + d_\lambda(x_0,\xi(t)) + d_\lambda^U(\xi(t),x) \leq \alpha + \delta + d_\lambda(x_0,x).$$

Letting δ go to zero yields the result. □

Example 3.2. It may happen that the pseudo-distance function $x \mapsto d_\lambda(x_0, x)$ is not a solution of the extension problem

$$(P_U) \qquad \min\left\{F(v,U) := \operatorname*{ess.sup}_{x\in U} H(x, Dv(x)) \;:\; v \in d_\lambda(x_0,.) + W^{1,\infty}(U) \cap \mathcal{C}_0(U)\right\},$$

where $x_0 \in \overline{\Omega} \setminus U$ and $U \subset\subset \Omega$ is connected and open: in that case, the inequality

$$d_\lambda^V(x_0,\cdot) \geq S^+(d_\lambda^V(x_0,\cdot), U)$$

is strict at some points of $\overline{U}$. As a consequence, $d_\lambda(x_0,.)$ is intuitively not a solution of some eikonal equation related to H. As an example, in $\mathbb{R}^2$ take $\Omega = B(0,2)$, $U = B(0,1)$ and

$$H(x,p) := \begin{cases} \frac{1}{2}|p| & if\ x \in \overline{U} \\ |p| & otherwise. \end{cases}$$

Now consider

$$(P_U) \qquad \min\left\{F(v,U) := \frac{1}{2}\|Dv\|_{L^\infty(U)} \;:\; v \in d_1(x_0,\cdot) + W^{1,\infty}(U) \cap \mathcal{C}_0(U)\right\}$$

where $x_0 = (1,0)$ belongs to $\overline{U}$. Then one has $F(d_1(x_0,\cdot),U) = \frac{2}{2} = 1$ and

$$\forall x,y \in \partial U \qquad |d_1(x_0,y) - d_1(x_0,x)| \leq \frac{\pi}{2}|y-x|$$

as well as

$$\forall x,y \in \overline{U} \qquad \forall \lambda \geq 0 \qquad d_\lambda^U(x,y) = 2\lambda|y-x|.$$

As a consequence of Theorem 2.11 one then infers that the minimal value $\inf(P_U)$ of (P_U) satisfies

$$\inf(P_U) \;\leq\; \frac{\pi}{4} \;<\; 1 \;=\; F(d_1(x_0,\cdot),U)$$

so that $d_1(x_0,\cdot)$ is not a solution of (P_U). Notice that the above fact is in contrast to the case $H(x,p) = |p|$, for which the geodesic distance $x \mapsto d_\lambda(x_0,x) = \lambda d(x_0,x)$ is an optimal solution of the extension problem (P_U) for any connected open set $U \subset\subset \Omega$ such that $x_0 \notin U$ (we of course assume here that $d(x_0,.)$ is finite on Ω, see Remark 2.6). Indeed, it is easy to verify that in that case the function $\lambda d(x_0,.)$ is a classical solution of the eikonal equation $|\nabla v| = \lambda$ in $\Omega \setminus \{x_0\}$, then derive that is is a solution of $-\Delta_\infty v = 0$ in $\Omega \setminus \{x_0\}$ and apply Theorems 3.1 and 3.2 of [12].

Definition 3.3. *We shall say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the Comparison with Distance Functions (noted CDF) from above in Ω if and only if for any connected open subset $V \subset\subset \Omega$, any $x_0 \in \overline{V}$, any $\lambda \geq 0$ and $\alpha \in \mathbb{R}$ the inequality*

$$u \leq d_\lambda^V(x_0,.) + \alpha \;\; on\ \partial(V \setminus \{x_0\})$$

implies

$$u \leq d_\lambda^V(x_0,.) + \alpha \;\; on\ \overline{V}.$$

Similarly, a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the CDF from below on Ω if and only if the inequality

$$u \geq -d_\lambda^V(.,x_0) + \alpha \;\; on\ \partial(V \setminus \{x_0\})$$

implies

$$u \geq -d_\lambda^V(.,x_0) + \alpha \;\; on\ \overline{V}.$$

Finally, a continuous function $u : \overline{\Omega} \to \mathbb{R}$ *satisfies the Comparison with Distance Functions on* Ω *if and only if it satisfies the CDF both from above and from below on* Ω.

Remark 3.4. The notion of Comparison with Distance Functions is a generalization of that of Comparison with Cones appearing in [12]. However, even for the classical case where $H : (x,p) \mapsto |p|$ is the euclidean norm the above notion is different from that introduced in [12]: indeed that paper deals with the comparison with the usual cones $x \mapsto \lambda|x - x_0|$, while our notion leads to the comparison with the cones $x \mapsto \lambda d^V(x_0, x)$ (where $d^V(\cdot,\cdot)$ is the usual geodesic distance in V, see Definition 2.3). This is overcomed in section 4, see Remark 4.4.

We now state and prove the main result of the paper.

Theorem 3.5. *Let* $u \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. *Then* u *is an absolute minimizer of*

$$(P) \qquad \min\left\{ F(v,\Omega) := \operatorname*{ess.sup}_{x\in\Omega} H(x, Dv(x)) \, : \, v \in g + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega) \right\}$$

if and only if $u = g$ *on* $\partial\Omega$ *and* u *satisfies the Comparison with Distance Functions on* Ω.

Proof. We first prove the *only if* part, that is if u is an absolute minimizer of (P) then it satisfies the CDF from above on Ω (the argument is the same for the CDF from below). Let thus V be an open, connected subset of Ω relatively compact in Ω, $x_0 \in \overline{V}$, $\lambda \geq 0$ and $\alpha \in \mathbb{R}$ be such that

$$u \leq d^V_\lambda(x_0, .) + \alpha \ \ on \ \partial(V \setminus \{x_0\}).$$

We shall assume that $d^V_\lambda(x_0,.)$ is not uniformly $+\infty$ on $\overline{V}$, so that is is Lipshitz continuous in V. Let $(\varepsilon_k)_k$ be a sequence of positive numbers decreasing to 0. Define $\alpha_k = \alpha + \varepsilon_k$ and observe that $(\alpha_k)_k$ decreases to α and

$$u < d^V_\lambda(x_0, .) + \alpha_k \ \ on \ \partial(V \setminus \{x_0\}). \tag{3.2}$$

Let $U_k := \{x \in V \, : \, u(x) > d^V_\lambda(x_0, .) + \alpha_k\}$. If U_k is empty, there is nothing to prove. Otherwise, we first claim that $\overline{U}_k \subset\subset (V \setminus \{x_0\})$. By contradiction, assume that the sequence $(x_n)_n$ in U_k converges to some $x \in \partial(V \setminus \{x_0\})$, then taking the liminf as n goes to $+\infty$ in $u(x_n) > d^V_\lambda(x_0, x_n) + \alpha_k$ yields

$$u(x) \geq d^V_\lambda(x_0, x) + \alpha_k,$$

and since $x \in \partial(V \setminus \{x_0\})$, this obviously contradicts (3.2).

Now since $\overline{U}_k \subset\subset (V \setminus \{x_0\})$ and u and d^V_λ are continuous on V, we have that U_k is open, and we may assume that it is connected (otherwise we consider a connected component). We then claim that

$$u \ \ \leq \ \ S^+(d^V_\lambda(x_0, .) + \alpha_k, U_k) \ \ on \ \overline{U}_k. \tag{3.3}$$

Indeed, $u = d^V_\lambda(x_0, .) + \alpha_k$ on ∂U_k, and since u is an absolute minimizer of (P), it is an optimal solution of

$$(P_{U_k}) \qquad \min\left\{ F(v, U_k) \, : \, v \in d^V_\lambda(x_0, .) + \alpha_k + W^{1,\infty}(U_k) \cap \mathcal{C}_0(U_k) \right\}.$$

Then (3.3) follows from Theorem 2.11, and Proposition 3.1 then allows to conclude that

$$u \leq d_\lambda^V(x_0,.) + \alpha_k \ \ on\ \overline{U}_k,$$

which obviously contradicts the definition of U_k which is then empty for all k. Letting k go to $+\infty$ we obtain

$$u \leq d_\lambda^V(x_0,.) + \alpha \ \ on\ V$$

which concludes the proof of the *only if* part.

We now turn to the *if* part: assume that $u = g$ on $\partial\Omega$ and u satisfies the CDF on Ω. Let V be an open subset of Ω with $V \subset\subset \Omega$, we must prove that u is an optimal solution of

$$(P_V) \qquad \min\left\{F(v,V)\,:\,v \in u + W^{1,\infty}(V) \cap \mathcal{C}_0(V)\right\}.$$

Thanks to Theorem 2.11, this is equivalent to show that

$$F(u,V) = \min(P_V) = \mu := \min\left\{\lambda\,:\,u(y) - u(x) \leq d_\lambda^V(x,y)\ for\ any\ x,y \in \partial V\right\}.$$

Let $x \in \partial V$, then by definition of μ one has

$$u(y) \leq u(x) + d_\mu^V(x,y)\ for\ any\ y \in \partial(V \setminus \{x\})$$

and since u satisfies the CDF from above this inequality holds for any $y \in \overline{V}$. Now let $y \in \overline{V}$, then we just obtained that

$$u(x) \geq u(y) - d_\mu^V(x,y)\ for\ any\ x \in \partial(V \setminus \{y\})$$

and since u satisfies the comparison with cones from below this inequality holds for any $x \in \overline{V}$. As a consequence, we get

$$u(y) - u(x) \leq d_\mu^V(x,y)\ for\ any\ x,y \in \overline{V}.$$

Proposition 2.10 then yields $F(u,V) \leq \mu$, which concludes the proof. □

Remark 3.6. The proof of the *if* part above is somewhat inspired from the argument of Proposition 2.1 in [1].

4. Comparison with Global Distance Functions

Dealing with distances which depend on the open set may be technically difficult. In this section, under some more regularity assumption on the supremand H (which still cover a very wide variety of possible supremands) we are able to simplify the results of section 3 by dealing with distances which do not depend anymore on the open set.

Definition 4.1. *We shall say that a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the Comparison with Global Distance Functions (noted CGDF) from above in Ω if for any connected open subset $V \subset\subset \Omega$, any $x_0 \in \overline{V}$, any $\lambda \geq 0$ and $\alpha \in \mathbb{R}$ the inequality*

$$u \leq d_\lambda^\Omega(x_0,.) + \alpha \ \ on\ \partial(V \setminus \{x_0\})$$

implies

$$u \leq d_\lambda^\Omega(x_0,.) + \alpha \ \ on\ \overline{V}.$$

Similarly, a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the CGDF from below on Ω if the inequality

$$u \geq -d_\lambda^\Omega(.,x_0) + \alpha \ \ on\ \partial(V \setminus \{x_0\})$$

implies

$$u \geq -d_\lambda^\Omega(.,x_0) + \alpha \ \ on\ \overline{V}.$$

Finally, a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the Comparison with Global Distance Functions on Ω if it satisfies the CGDF both from above and from below on Ω.

In order to make the link between the Comparison with Global Distance Functions and the absolute minimizers, we assume in the rest of this section that the following regularity property on H holds:

(D) For all $\lambda > \mu \geq 0$ and $V \subset\subset \Omega$ there exists $\alpha > 0$ such that

$$\forall x \in V \qquad \{\ H(x,\cdot) < \mu\} + B(0,\alpha)\ \subset\ \{\ H(x,\cdot) < \lambda\}$$

Remark 4.2. It is easily infered from this assumption that for any connected open set $V \subset\subset \Omega$ and for any $\lambda > \mu \geq 0$ one has

$$\forall x\ \neq y \in V \qquad d_\mu^V(x,y) < d_\lambda^V(x,y).$$

We now turn to the main theorem of this section.

Theorem 4.3. *Assume that (D) holds, and let $u \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then u is an absolute minimizer of*

$$(P) \qquad \min\left\{F(v,\Omega) := \operatorname*{ess.sup}_{x\in\Omega} H(x,Dv(x))\ :\ v \in g + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)\right\}$$

if and only if $u = g$ on $\partial\Omega$ and u satisfies the Comparison with Global Distance Functions on Ω.

Remark 4.4. As the proof of the above theorem shows, the CGDF characterization is obtained via local arguments: every computation is made in some $V \subset\subset \Omega$. One could even therefore replace the pseudo-distances d_λ^Ω by the pseudo-distances $d_\lambda^{\mathbb{R}^N}$ whenever H is defined on $\mathbb{R}^N \times \mathbb{R}^N$, with assumption (D) still holding on Ω. One may in particular apply this to the case where H does not depend on x: for example, one thus recovers the usual comparison with cones for the special case $H(x,p) = |p|$.

Proof of Theorem 4.3. For the *only if* part, we just notice that the corresponding proof of Theorem 3.5 still holds when d_λ^V is replaced with d_λ^Ω.

Let us turn to the *if* part: assume that $u = g$ on $\partial\Omega$ and u satisfies the CGDF on Ω. Let V be a connected open subset of Ω with $V \subset\subset \Omega$, we must prove that u is an optimal solution of

$$(P_V) \qquad \min\left\{F(v,V)\ :\ v \in u + W^{1,\infty}(V) \cap \mathcal{C}_0(V)\right\}.$$

By Theorem 2.11, this is equivalent to show that

$$F(u,V) = \min(P_V) = \mu := \min\left\{\lambda\ :\ u(y) - u(x) \leq d_\lambda^V(x,y)\ for\ any\ x,y \in \partial V\right\}.$$

By contradiction, assume that $\Lambda := F(u,V) > \mu$. Then there exists $x_0 \in V$ such that u is differentiable at x_0 and

$$H(x_0, \nabla u(x_0)) = \lambda_0 > \mu. \tag{4.1}$$

We first claim that for $\lambda = (\mu + \lambda_0)/2$ there exists $x_{+\infty} \in \partial V$ such that

$$u(x_{+\infty}) - u(x_0) \quad \geq \quad d_\lambda^V(x_0, x_{+\infty}) \tag{4.2}$$

where $\lambda = (\mu + \lambda_0)/2$. Let $M > 1 > \alpha > 0$ be such that

$$B(0, \alpha) \subset \{ H(x, \cdot) < \nu \} \subset B(0, M)$$

for all $x \in V$ and $\nu \in [\mu, \Lambda]$.

We define by induction the decreasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ by $\lambda_{n+1} := (\mu+\lambda_n)/2$ for any $n \geq 0$. Then we infer from (4.1) that there exists x_0' such that $|x_0'-x_0| \leq \frac{\alpha}{2M^2} dist(x_0, \partial V)$ and

$$u(x_0') > u(x_0) + d^V_{\lambda_1}(x_0, x_0').$$

Indeed, one would otherwise have $u(x) \leq u(x_0)+d^V_{\lambda_1}(x_0, x)$ for any $x \in B(x_0, \frac{\alpha}{2M^2} dist(x_0, \partial V))$, then Proposition 2.10 would imply $H(x_0, \nabla u(x_0)) \leq \lambda_1$, which contradicts (4.1).

Let us notice that since $|x_0'-x_0| \leq \frac{\alpha}{M} dist(x_0, \partial V)$, Lemma B.4 yields that $d^V_{\lambda_1}(x_0, x_0') = d^\Omega_{\lambda_1}(x_0, x_0')$. We may now apply Lemma 4.5 below to build by induction starting from x_0 a sequence $(x_n)_{n\in\mathbb{N}}$ in V such that

- for any $n \in \mathbb{N}$, one has
$$\frac{\alpha^2}{2M^2} dist(x_n, \partial V) \leq |x_{n+1} - x_n| \leq \frac{\alpha}{2M} dist(x_n, \partial V),$$
- for any $n \in \mathbb{N}$, one has
$$u(x_{n+1}) \geq u(x_n) + d^\Omega_{\lambda_{n+1}}(x_n, x_{n+1}),$$
- for any $n \in \mathbb{N}$, $x_n \in V$ and there exists $x_n' \in V$ such that
$$|x_n' - x_n| \leq \frac{\alpha^2}{2M^2} dist(x_n, \partial V) \qquad \textit{and} \qquad u(x_n') > u(x_n) + d^\Omega_{\lambda_{n+1}}(x_n, x_n').$$

Applying once again Lemma B.4 yields that in the preceding, $d^V_{\lambda_n} = d^\Omega_{\lambda_n}$, so that the sequence $(x_n)_{n\in\mathbb{N}}$ is such that for any $n \geq 0$, one has

$$|x_{n+1} - x_n| \geq \frac{\alpha^2}{2M^2} dist(x_n, \partial V) \qquad \textit{and} \qquad u(x_{n+1}) \geq u(x_n) + d^V_{\lambda_{n+1}}(x_n, x_{n+1}).$$

This yields that for any $n \in \mathbb{N}$

$$u(x_{n+1}) - u(x_n) \ \geq\ d^V_\lambda(x_n, x_{n+1}) \ \geq\ \frac{\alpha^3}{2M^2} dist(x_n, \partial V).$$

We now conclude as in the proof of Theorem 3.2 of [12]: for any $n \in \mathbb{N}$ one has

$$u(x_{n+1}) - u(x_0) \ \geq\ \sum_{i=0}^{n} d^V_\lambda(x_i, x_{i+1}) \ \geq\ \frac{\alpha^3}{2M^2} \sum_{i=0}^{n} dist(x_i, \partial V) \geq 0.$$

Since u is continuous on $\overline{\Omega}$, it is bounded on $\overline{V}$, and thus $dist(x_n, \partial V) \to 0$ as $n \to +\infty$. Let $x_{+\infty}$ be a cluster point of $(x_n)_n$, then $x_{+\infty} \in \partial V$ and taking the liminf as n goes to $+\infty$ in

$$u(x_{n+1}) - u(x_0) \ \geq\ \sum_{i=0}^{n} d^V_\lambda(x_i, x_{i+1}) \ \geq\ d^V_\lambda(x_0, x_{n+1})$$

one gets (4.2), which proves the claim.

A similar argument (which relies on the use of Lemma 4.6) yields the existence of some $x_{-\infty} \in \partial V$ such that

$$u(x_0) - u(x_{-\infty}) \;\geq\; d^V_\lambda(x_{-\infty}, x_0). \tag{4.3}$$

We finally infer from (4.2) and (4.3) that for the points $x_{+\infty}, x_{-\infty} \in \partial V$ it holds

$$u(x_{+\infty}) - u(x_{-\infty}) \;\geq\; d^V_\lambda(x_0, x_{+\infty}) + d^V_\lambda(x_{-\infty}, x_0) \;\geq\; d^V_\lambda(x_{-\infty}, x_{+\infty}).$$

By Remark 4.2, we then get

$$u(x_{+\infty}) - u(x_{-\infty}) \;\geq\; d^V_\lambda(x_{-\infty}, x_{+\infty}) \;>\; d^V_\mu(x_{-\infty}, x_{+\infty}),$$

which contradicts the definition of μ. □

The following Lemma is inspired from Lemmas 2.4 and 3.3 of [12]. Notice that in its proof, we only use that u satisfies the CGDF from above. For the notations and hypotheses, we refer to Theorem 4.3.

Lemma 4.5. *Assume that $x \in V$ is such that there exist $\nu \in]\mu, \Lambda]$ and $x' \in V$ such that $|x' - x| \leq \frac{\alpha^2}{2M^2} dist(x, \partial V)$ and*

$$u(x') > u(x) + d^\Omega_\nu(x, x').$$

Then for any $\theta \in]\mu, \nu[$ there exist y and y' in V such that

- $\frac{\alpha^2}{2M^2} dist(x, \partial V) \leq |y - x| \leq \frac{\alpha}{2M} dist(x, \partial V)$,
- $u(y) \geq u(x) + d^\Omega_\theta(x, y)$,
- $|y' - y| \leq \frac{\alpha^2}{2M^2} dist(y, \partial V)$ *and* $u(y') > u(y) + d^\Omega_\theta(y, y')$.

Proof. Set $R := \frac{\alpha^2}{2M} dist(x, \partial V)$, let $\theta \in]\mu, \nu[$, $\theta' \in]\theta, \nu[$ and define

$$a := \max\{u(z) - d^\Omega_{\theta'}(x, z) \,:\, z \textit{ such that } d^\Omega_{\theta'}(x, z) \leq R\}.$$

By the definition of R, if $d^\Omega_{\theta'}(x, z) \leq R$ then

$$\alpha|z - x| \leq d^\Omega_{\theta'}(x, z) \leq R = \frac{\alpha^2}{2M} dist(x, \partial V)$$

so that $z \in V$. Moreover, since x' is such that

$$d^\Omega_{\theta'}(x, x') \leq M|x' - x| \leq M\frac{\alpha^2}{2M^2} dist(x, \partial V) = R,$$

one has $a \geq u(x') - d^\Omega_{\theta'}(x, x') \geq u(x') - d^\Omega_\nu(x, x') > u(x)$. Since u satisfies the CGDF from above on Ω and $d^\Omega_{\theta'}(x, \cdot)$ is continuous on V, one has

$$a = \max\{u(z) - d^\Omega_{\theta'}(x, z) \,:\, z = x \textit{ or } z \textit{ such that } d^\Omega_{\theta'}(x, z) = R\},$$

and we infer from $d^\Omega_{\theta'}(x, x') \leq R$ that

$$a = \max\{u(z) - d^\Omega_{\theta'}(x, z) \,:\, z \textit{ such that } d^\Omega_{\theta'}(x, z) = R\} > u(x).$$

Let now y be such that $d^\Omega_{\theta'}(x, y) = R$ and $u(y) - d^\Omega_{\theta'}(x, y) = a$. Then $y \in V$, and one has

$$M|y - x| \geq d^\Omega_{\theta'}(x, y) = R \quad \textit{so that} \quad |y - x| \geq \frac{\alpha^2}{2M^2} dist(x, \partial V)$$

as well as

$$\alpha|y-x| \le d^{\Omega}_{\theta'}(x,y) = R \quad \textit{so that} \quad |y-x| \le \frac{\alpha}{2M} dist(x,\partial V)$$

and

$$u(y) = d^{\Omega}_{\theta'}(x,y) + a > d^{\Omega}_{\theta'}(x,y) + u(x) \ge d^{\Omega}_{\theta}(x,y) + u(x).$$

Now, set $r = \frac{\alpha^3}{2M^2} dist(y,\partial V)$. By Lemma B.5, for any $\varepsilon \in]0,1[$ there exists y_ε such that $|y_\varepsilon - y| = \varepsilon \min\{\frac{r}{M}, \frac{|y-x|}{2}\}$ and

$$d^{\Omega}_{\theta'}(x,y) \;=\; d^{\Omega}_{\theta'}(x,y_\varepsilon) + d^{\Omega}_{\theta'}(y_\varepsilon,y).$$

As a consequence, $d^{\Omega}_{\theta'}(x,y_\varepsilon) \le R$ so that

$$\begin{aligned} u(y_\varepsilon) - d^{\Omega}_{\theta'}(x,y_\varepsilon) \;\le\; a &= u(y) - d^{\Omega}_{\theta'}(x,y) \\ &= u(y) - \left(d^{\Omega}_{\theta'}(x,y_\varepsilon) + d^{\Omega}_{\theta'}(y_\varepsilon,y)\right). \end{aligned}$$

We thus have

$$u(y_\varepsilon) \le u(y) - d^{\Omega}_{\theta'}(y_\varepsilon,y). \tag{4.4}$$

Take $\theta'' \in]\theta,\theta'[$ and define

$$b := \max\{u(z) - d^{\Omega}_{\theta''}(y_\varepsilon,z) \;:\; z \textit{ such that } d^{\Omega}_{\theta''}(y_\varepsilon,z) \le r\}.$$

We notice that $d^{\Omega}_{\theta''}(y_\varepsilon,y) \le M|y_\varepsilon - y| \le r$, and by (4.4) and Remark (4.2) we have

$$b \ge u(y) - d^{\Omega}_{\theta''}(y_\varepsilon,y) > u(y) - d^{\Omega}_{\theta'}(y_\varepsilon,y) \ge u(y_\varepsilon).$$

The same arguments as above thus yield the existence of some y^ε such that

$$d^{\Omega}_{\theta''}(y_\varepsilon,y^\varepsilon) = r \qquad \textit{and} \qquad u(y^\varepsilon) - d^{\Omega}_{\theta''}(y_\varepsilon,y^\varepsilon) = b > u(y_\varepsilon).$$

Let y' be a cluster point of the family $(y^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$, then by passing to the limit in the above relations, one infers

$$d^{\Omega}_{\theta''}(y,y') = r \qquad \textit{and} \qquad u(y') - d^{\Omega}_{\theta''}(y,y') \ge u(y).$$

Then one has

$$|y'-y| \le \frac{1}{\alpha} d^{\Omega}_{\theta''}(y,y') = \frac{r}{\alpha} \le \frac{\alpha^2}{2M^2} dist(y,\partial V)$$

Finally using once again Remark (4.2) we get

$$u(y') - d^{\Omega}_{\theta}(y,y') > u(y') - d^{\Omega}_{\theta''}(y,y') \ge u(y)$$

which concludes the proof of the Lemma. □

An analogue of the above Lemma holds, if we require only that u satisfies the CGDF from below on Ω.

Lemma 4.6. *Assume that $x \in V$ is such that there exist $\nu \in]\mu,\Lambda]$ and $x' \in V$ such that $|x'-x| \le \frac{\alpha^2}{2M^2} dist(x,\partial V)$ and*

$$u(x) > u(x') + d^{\Omega}_{\nu}(x',x).$$

Then for any $\theta \in]\mu,\nu[$ there exist y and y' in V such that

- $\frac{\alpha^2}{2M^2} dist(x,\partial V) \le |y-x| \le \frac{\alpha}{2M} dist(x,\partial V)$,
- $u(x) \ge u(y) + d^{\Omega}_{\theta}(y,x)$,

- $|y'-y| \leq \frac{\alpha^2}{2M^2} dist(y,\partial V)$ *and* $u(y) > u(y') + d_\theta^\Omega(y',y)$.

5. Stability of absolute minimizers with respect to Γ-convergence

In this part, we show that the notion of absolute minimizer is stable with respect to Γ-convergence (see Theorem 5.1 below) and then apply this result to the case of the homogenization in L^∞ of supremal functionals. We recall that when X is a metric space, a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge to F in X if

$$\forall x \in X \qquad F(x) = \Gamma - \liminf F_n(x) = \Gamma - \limsup F_n(x),$$

where

$$\Gamma - \liminf F_n(x) = \inf \left\{ \liminf F_n(x_n) : \quad x_n \to x \ in \ X \right\}$$
$$\Gamma - \limsup F_n(x) = \inf \left\{ \limsup F_n(x_n) : \quad x_n \to x \ in \ X \right\}.$$

We refer to [13] for an introduction to the theory of Γ-convergence. We now state the stability of absolute minimizers with respect to this notion of convergence.

Theorem 5.1. *Assume that for any* $n \in \mathbb{N}$*, the function* $u_n \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ *is an absolute minimizer of the problem*

$$(P_n) \qquad \min \left\{ \operatorname*{ess.sup}_{x\in\Omega} H_n(x, Dv(x)) \ : \ v \in u_n + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega) \right\},$$

and that the sequence $(u_n)_{n\in\mathbb{N}}$ *converges uniformly on* $\overline{\Omega}$ *to some function* $u_\infty \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$*. For any* $n \in \mathbb{N} \cup \{+\infty\}$ *and relatively compact open subset* $U \subset \Omega$ *with boundary of class* $\mathcal{C}^2$*, we define the supremal functional* $F_n(.,U)$ *on* $\mathcal{C}(\overline{U})$ *by*

$$F_n(v,U) := \begin{cases} \operatorname*{ess.sup}_{x\in U} H_n(x, Dv(x)) & \text{if } v \in W^{1,\infty}(U) \cap \mathcal{C}(\overline{U}), \\ +\infty & \text{otherwise,} \end{cases}$$

where the supremand H_n *satisfies conditions* (A) *and* (C) *for any* $n \in \mathbb{N} \cup \{+\infty\}$*, and the family* $\{H_n\}_{n\in\mathbb{N}\cup\{+\infty\}}$ *is uniformly equicoercive on* $\Omega \times \mathbb{R}^N$*, i.e.*

$$\forall \lambda \geq 0 \quad \exists M > 0 \quad \forall n \in \mathbb{N} \cup \{+\infty\} \quad \forall (x,p) \qquad H_n(x,p) \leq \lambda \Rightarrow |p| \leq M. \quad (5.1)$$

Suppose that for any such open subset U *the sequence* $(F_n(.,U))_n$ Γ*-converges in* $\mathcal{C}(\overline{U})$ *to the supremal functional* $F_\infty(.,U)$*. Then the function* u_∞ *is an absolute minimizer of*

$$(P_\infty) \qquad \min \left\{ \operatorname*{ess.sup}_{x\in\Omega} H_\infty(x, Dv(x)) \ : \ v \in u_\infty + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega) \right\}.$$

We first prove the following Lemma, which in our opinion has an interest by itself. For a connected open subset $V \subset \Omega$, $\lambda \geq 0$ and $n \in \mathbb{N} \cup \{+\infty\}$, we shall denote by $d_{\lambda,n}^V$ the pseudo-distance in V associated with the supremand H_n.

Lemma 5.2. *Under the assumptions of Theorem 5.1, for any* $U \subset\subset \Omega$ *with boundary of class* $\mathcal{C}^2$*, any* $\mu \geq 0$ *and* $x \in \overline{U}$*, there exists a sequence of non-negative real numbers* $(\mu_n)_n$ *such that*

$$d_{\mu_n,n}^U(x,.) \to d_{\mu,\infty}^U(x,.) \quad in \ \mathcal{C}(\overline{U}). \quad (5.2)$$

Proof. We infer from Lemma B.3 that for any $y \in \overline{U}$ one has

$$\begin{aligned} d^U_{\mu,\infty}(x,y) &= \sup\{v(y)-v(x) \ : \ v \in W^{1,\infty}(U)\cap \mathcal{C}(\overline{U}),\ H_\infty(.,Dv) \le \mu \text{ a.e. on } V\} \\ &= \sup\{v(y)-v(x) \ : \ F_\infty(v,U) \le \mu\}. \end{aligned}$$

Since $(F_n(.,U))_n$ Γ-converges in $\mathcal{C}(\overline{U})$ to $F_\infty(.,U)$, there exists a sequence of functions $v_n \in W^{1,\infty}(U)\cap\mathcal{C}(\overline{U})$ converging uniformly in $\overline{U}$ to $d^U_{\mu,\infty}(x,.)$ and such that

$$F_n(v_n,U) \to F_\infty(d^U_{\mu,\infty}(x,.),U) \le \mu$$

where the last inequality follows from Proposition 2.10. For any n we set $F_n(v_n,U)=\mu_n$ and we prove (5.2) for the sequence $(\mu_n)_n$. It follows again from Lemma B.3 that for any n and $y \in \overline{U}$ one has

$$\begin{aligned} d^U_{\mu_n,n}(x,y) &= \sup\{v(y)-v(x) \ : \ F_n(v,U) \le \mu_n\} \\ &\ge v_n(y)-v_n(x), \end{aligned}$$

thus letting n go to $+\infty$ yields

$$\forall y \in \overline{U} \qquad \liminf_{n\to+\infty} d^U_{\mu_n,n}(x,y) \ge d^U_{\mu,\infty}(x,y).$$

We also notice that by the uniform equicoercivity assumption (5.1), the regularity of ∂U and the fact that $d^U_{\mu_n,n}(x,x)=0$ for any n, the family $(d^U_{\mu_n,n}(x,.))_n$ is uniformly bounded and equicontinuous on $\overline{U}$ and thus we may assume without loss of generality that it converges in $\mathcal{C}(\overline{U})$ to some function w. We infer from the Γ-convergence of $F_n(.,U)$ that

$$F_\infty(w,U) \ \le\ \liminf_{n\to+\infty} F_n(d^U_{\mu_n,n}(x,.),U) \ \le\ \liminf_{n\to+\infty}\mu_n \le \mu.$$

Then for any y in $\overline{U}$ we have

$$\begin{aligned} \limsup_{n\to+\infty} d^U_{\mu_n,n}(x,y) \ = \ w(y) \ &= \ w(y)-w(x) \\ &\le \ \sup\{v(y)-v(x) \ : \ F_\infty(v,U)\le\mu\} \ = \ d^U_{\mu,\infty}(x,y) \end{aligned}$$

where the last inequality follows by Proposition 2.10 and this concludes the proof. □

Proof of Theorem 5.1. Thanks to theorem 3.5, it is sufficient to prove that u_∞ satisfies the CDF associated with H_∞. We only prove that u_∞ satisfies the CDF from above, the argument being similar for the comparison from below. Let then $x_0 \in \overline{V}$, $\lambda \ge 0$ and $\alpha \in \mathbb{R}$ be such that

$$u_\infty \le d^V_{\lambda,\infty}(x_0,.)+\alpha \ \text{ on } \partial(V\setminus\{x_0\}).$$

Let $\varepsilon > 0$ and set $W = \{x \in V \ : \ u_\infty > d^V_{\lambda,\infty}(x_0,.)+\alpha+\varepsilon\}$. If W is empty, there is nothing to prove, otherwise the same arguments as in the proof of Theorem 3.5 yield that W is open and $W \subset\subset V\setminus\{x_0\}$. We may also assume without loss of generality that W is connected. Thus there exists $U \subset\subset V\setminus\{x_0\}$ open, connected, with $\mathcal{C}^2$ boundary and containing $\overline{W}$. Then the function $d^V_{\lambda,\infty}(x_0,.)$ is continuous on $\overline{U}$, and we have

$$u_\infty(y) \ \le\ \alpha+\varepsilon+d^V_{\lambda,\infty}(x_0,y) \ = \ \alpha+\varepsilon+S^+(d^V_{\lambda,\infty}(x_0,.),U)(y)$$

for any $y \in \partial U$. We now infer from the definition of $S^+(d^V_{\lambda,\infty}(x_0,.),U)$ that

$$u_\infty(y) \ \le \ \alpha+\varepsilon+d^V_{\lambda,\infty}(x_0,x)+d^U_{\mu,\infty}(x,y) \tag{5.3}$$

for any $x, y \in \partial U$ and with

$$\mu = \min\left\{\sigma \,:\, d^V_{\lambda,\infty}(x_0,y) - d^V_{\lambda,\infty}(x_0,x) \le d^U_{\sigma,\infty}(x,y) \ \textit{for any}\ x,y \in \partial U\right\}.$$

Let us now fix $x \in \partial U$ in (5.3), we aim to show that this inequality holds for any y in $\overline{U}$. We infer from Lemma 5.2 that there exists a sequence $(\mu_n)_n$ such that

$$d^U_{\mu_n,n}(x,.) \to d^U_{\mu,\infty}(x,.) \quad \textit{in}\ \mathcal{C}(\overline{U}).$$

Let $\delta > 0$, we get from (5.3) that for n large enough

$$\forall y \in \partial U \qquad u_n(y) \ \le\ \alpha + \varepsilon + \delta + d^V_{\lambda,\infty}(x_0,x) + d^U_{\mu_n,n}(x,y) \tag{5.4}$$

Since for all n the function u_n satisfies the CDF property, the inequality (5.4) holds for any $y \in \overline{U}$. Letting n go to ∞ in (5.4), we get that

$$\forall y \in \overline{U} \qquad u_\infty(x) \le \alpha + \varepsilon + \delta + d^V_{\lambda,\infty}(x_0,x) + d^U_{\mu,\infty}(x,y).$$

We let δ go to 0 and take the infimum on $x \in \partial U$ to get that for any $y \in \overline{U}$ one has

$$\begin{aligned} u_\infty(y) &\le \alpha + \varepsilon + \inf\{d^V_{\lambda,\infty}(x_0,x) + d^U_{\mu,\infty}(x,y) \ :\ x \in \partial U\} \\ &= \alpha + \varepsilon + S^+(d^V_{\lambda,\infty}(x_0,.),U)(y). \end{aligned}$$

Applying Proposition 3.1 yields

$$\forall y \in \overline{U} \qquad u_\infty(y) \le \alpha + \varepsilon + d^V_{\lambda,\infty}(x_0,y),$$

which contradicts the definitions of U and W. □

We now turn to the application of the above result to an homogenization problem. For any positive ε and V open subset of Ω, let the supremal functional $F_\varepsilon(.,V)$ be defined on $\mathcal{C}(\overline{V})$ by

$$F_\varepsilon(v,V) := \begin{cases} \operatorname*{ess.sup}_{x\in\Omega} H(\frac{x}{\varepsilon}, Dv(x)) & \text{if } v \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V}), \\ +\infty & \text{otherwise,} \end{cases}$$

and let $F_{hom}(.,V)$ be given by

$$F_{hom}(v,V) := \begin{cases} \operatorname*{ess.sup}_{x\in\Omega} H_{hom}(Dv(x)) & \text{if } v \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V}), \\ +\infty & \text{otherwise,} \end{cases}$$

where for any $p \in \mathbb{R}^N$ one has

$$H_{hom}(p) := \inf\left\{ \operatorname*{ess.sup}_{x\in(0,1)^N} H(x, p + Dw(x)) \ :\ w \in W^{1,\infty}_{\#}((0,1)^N) \cap \mathcal{C}((0,1)^N) \right\}.$$

The following classical assumptions are made on the supremand H:

(A') $H : \mathbb{R}^N \times \mathbb{R}^N \to [0,+\infty]$ is lower semi-continuous, $(0,1)^N$-periodic in the first variable and level-convex in the second, and $H(.,0) = 0$.

(B') H satisfies the growth condition: $\alpha(|p|) \le H(x,p) \le \beta(|p|)$ for any $(x,p) \in \mathbb{R}^N \times \mathbb{R}^N$, where $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}_+$ are increasing functions, $\alpha(t) \to +\infty$ as $t \to +\infty$ and β is locally bounded.

(E) H satisfies the continuity condition: for any $M > 0$ there exists a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(t) \to 0$ as $t \to 0^+$ and

$$\forall x \in (0,1)^N \quad \forall p, \eta \in B(0,M) \qquad |H(x,p) - H(x,\eta)| \leq \omega(|p-\eta|).$$

Then the following holds.

Theorem 5.3. *Assume that for any $\varepsilon > 0$, the function $u_\varepsilon \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is an absolute minimizer of the problem*

$$(P_\varepsilon) \qquad \min\left\{F_\varepsilon(v,\Omega) \,:\, v \in u_\varepsilon + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)\right\}$$

and that $u_0 \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is a cluster point (as $\varepsilon \to 0$) in $\mathcal{C}(\overline{\Omega})$ of the family $(u_\varepsilon)_{\varepsilon>0}$. Then u_0 is an absolute minimizer of the homogenized problem

$$(P_{hom}) \qquad \min\left\{F_{hom}(v,\Omega) \,:\, v \in u_0 + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)\right\}.$$

Proof. We first notice that thanks to hypotheses (A') and (B'), the family of supremands $(H(\frac{\cdot}{\varepsilon},.))_{\varepsilon>0}$ obviously satisfies conditions (A), (C) and (5.1). To apply theorem 5.1, it thus remains to prove that for any open subset $V \subset\subset \Omega$ with $\mathcal{C}^2$ boundary the family $(F_\varepsilon(.,V))$ Γ-converges to $F_{hom}(.,V)$ on $\mathcal{C}(\overline{V})$ as $\varepsilon \to 0$: this follows from Theorem 5.2 of [6]. □

6. Absolutely minimizing Lipschitz extensions in length spaces

In this section we show that the principle of Comparison with Distance Functions also characterizes the absolutely minimizing Lipschitz extensions in a length space (X,d). A metric space (X,d) is said a length space if it is arcwise-connected and the distance of any two points coincide with the infimum of the length of continuous arcs joining them. More precisely, if $x,y \in X$ and if we denote by $\mathrm{path}_X(x,y)$ the set of continuous maps $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, then the length $l(\gamma)$ of γ is given by

$$l(\gamma) := \sup\left\{\sum_{i=0}^{k} d(\gamma(t_i),\gamma(t_{i+1})) \;:\; 0 = t_0 \leq \ldots \leq t_k = 1,\ k \geq 1\right\}$$

and (X,d) is a length space if for any $x,y \in X$ one has

$$d(x,y) = \inf\{l(\gamma) \;:\; \gamma \in \mathrm{path}_X(x,y)\}$$

The category of length spaces includes Riemannian manifolds, Carnot-Caratheodory spaces, as well as more general spaces (see [1] for more on this notion).

We first recall the definition of an Absolutely minimizing Lipschitz extension in a length space as given in §9 of [1].

Definition 6.1. *Let (X,d) be a length space, $A \subset X$ and $u : A \to \mathbb{R}$, then we set*

$$Lip(u,A) \;:=\; \inf\{k \;:\; u(y) - u(x) \leq kd(x,y) \ for\, all\, x,y \in A\}.$$

Let V be a proper open subset of X, then a Lipschitz continuous function $u : V \to \mathbb{R}$ is said to be an Absolutely Minimizing Lipschitz function (noted AML) on V if for all open subset U of V one has

$$Lip(u,U) = Lip(u,\partial U).$$

In the above definition, proper means that $V \notin \{\emptyset, X\}$, and since X is connected this implies that $\partial V \neq \emptyset$. We now extend the notion of Comparison with Distance Functions to this setting.

Definition 6.2. *Let (X, d) be a length space and V a proper open subset of X. A function $u : \overline{V} \to \mathbb{R}$ satisfies the Comparison with Distance Functions from above on V if for any open subset U of V, any $x_0 \in X$, any $a \geq 0$ and $b \in \mathbb{R}$, the inequality*

$$u \leq ad(x_0, .) + b \ \text{ on } \partial(U \setminus \{x_0\})$$

implies

$$u \leq ad(x_0, .) + b \ \text{ on } \overline{U}.$$

Similarly, a continuous function $u : \overline{\Omega} \to \mathbb{R}$ satisfies the CDF from below on Ω if and only if the inequality

$$u \geq -ad(x_0, .) + b \ \text{ on } \partial(U \setminus \{x_0\})$$

implies

$$u \geq -ad(x_0, .) + b \ \text{ on } \overline{U}.$$

Finally, u satisfies the CDF on V if and only if it satisfies the CDF both from above and from below on V.

Remark 6.3. We point out that in the above definition the parameter a is non-negative, whereas in the classical definition (for example see Definition 2.2 in [1]) a may take negative values. This is inspired from Definition 3.3 where the parameter λ is necessarily non-negative. This difference allows to show that with the above definition, the comparison with cones is still a characterization for the AML property in this general length space setting, and rules out the problem arising in Example 9.2 of [1]. Indeed, in that example the authors consider the case where X is the unit sphere of $\mathbb{R}^3$ equipped with the geodesic distance, x_0 is the north pole and V the southern hemisphere. Then the constant function $\frac{\pi}{2}$ is of course an AML in X, and is equal to the cone $d_1(x_0, .)$ on ∂V. One doesn't have $\frac{\pi}{2} \geq d_1(x_0, .)$ in V, while $\frac{\pi}{2} \leq d_1(x_0, .)$ holds in V. The classical definition for the comparison with cones would ask the two inequalities to hold (and thus fails to characterize the AML property), whereas Definition 6.2 only asks for the second inequality to hold.

We now state the main theorem of this part.

Theorem 6.4. *Let V be a proper open subset of X, then $u : \overline{V} \to \mathbb{R}$ is an AML in V if and only if u enjoys the Comparison with Distance Functions property.*

Proof. The *only if* part. We only prove that u satisfies the CDF from above on V. Let $U \subset V$ be open, $x_0 \in X$, $a \geq 0$ and $b \in \mathbb{R}$ and assume that

$$u \leq ad(x_0, .) + b \ \text{ on } \partial(U \setminus \{x_0\}).$$

Assume, by contradiction, that the set $A := \{x \in U \ : \ u(x) > ad(x_0, x) + b\}$ is not empty. Then A is open and on ∂A we have $u(x) = ad(x_0, x)+b$. The triangular inequality implies that $Lip(ad(x_0, .) + b, \partial A) = \alpha \leq a$, and the maximal Lipschitz extension u^+ of this distance function inside A is

$$u^+ : x \mapsto \inf_{y \in \partial A} \{ad(x_0, y) + b + \alpha d(y, x)\}.$$

Since u is an AML on V, one has $u \le u^+$ on $\overline{A}$. Proceeding as in the proof of Proposition 3.1, we get that $u^+ \le ad(x_0, .) + b$ on A. Indeed, for any $x \in A$ one has

$$u^+(x) \quad \le \quad \inf_{y \in \partial A} \{ad(x_0, y) + b + ad(y, x)\}. \tag{6.1}$$

Let $\delta > 0$, there exists $\gamma \in \mathrm{path}_X(x_0, x)$ such that $d(x_0, x) \ge l(\gamma) - \delta$, then taking $t_\delta \in [0, 1]$ such that $\tilde{x} = \gamma(t_\delta) \in \partial A$, one gets $d(x_0, x) \ge d(x_0, \tilde{x}) + d(\tilde{x}, x) - \delta$. Since $a \ge 0$, we infer by taking $y = \tilde{x}$ in (6.1) that $u^+(x) \le ad(x_0, x) + b + a\delta$. Letting δ go to 0, we finally get the contradiction.

The *if* part follows line by line the proof of Proposition 2.1 in [1]. □

Remark 6.5. We point out that in the preceding proof, the geometric idea of the first implication is again the idea of Proposition 3.1, and that this idea is easily adapted to this length space setting.

Remark 6.6. The previous theorem shows that even in an ambient space in which cones do not satisfy comparison with cones (for example a sphere or any other manifold with non trivial cut-locus) the CDF property still characterizes the absolutely minimizing Lipschitz extension. Then the CDF property provides, in some sense, a structure-free criterium for absolute minimality.

Appendix A. Finsler metrics and related questions

A Finsler metric on a connected open subset Ω of $\mathbb{R}^n$ is a Borel-measurable function $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ such that $\varphi(x, \cdot)$ is positively 1-homogeneous for all $x \in \Omega$ and convex for $\mathcal{L}^n$ a.e. $x \in \Omega$. We refer to [2] for an advanced introduction. We also refer to the papers [14] for more complete versions of Propositions A.1 and A.2 (respectively Theorems 3.3 and 3.7 therein).

Given a positive constant β we set

$$\mathcal{M}_\beta = \{\varphi \text{ Finsler metric in } \Omega \text{ s.t. } \ \varphi(x, q) \le \beta|q| \text{ in } \Omega \times \mathbb{R}^n\}.$$

Then to each $\varphi \in \mathcal{M}_\beta$ one can associate (as we did in section 2) a pseudo distance $d_\varphi : \Omega \times \Omega \to \mathbb{R}_+$ through the formula

$$d_\varphi(x, y) = \inf \left\{ \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))dt \ : \ \gamma \in \mathrm{path}(x, y) \right\}.$$

There is in the literature another way to associate a distance to a Finsler metric $\varphi \in \Omega$ and it consists of a $\sup - \inf$ operation. We denote by $\mathcal{N}$ the set of subsets N of Ω with Lebesgue measure $\mathcal{L}^n(N) = 0$. A Lipschitz curve $\gamma : [0, 1] \to \Omega$ is said transversal to N if $\mathcal{H}^1(\gamma([0, 1]) \cap N) = 0$. Then we define

$$d^\varphi(x, y) = \sup_{N \in \mathcal{N}} \inf \left\{ \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))dt \ : \ \gamma \in \mathrm{path}(x, y) \text{ and } \gamma \text{ transversal to } N \right\}.$$

We now introduce the polar of φ, $\varphi^0 : \Omega \times \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ which is defined as follows:

$$\varphi^0(x, p) := \sup\{p \cdot q \ : \ \varphi(x, q) \le 1\}.$$

We then have the following:

Proposition A.1. *The following inequality holds for any $x, y \in \Omega$*

$$\sup\{u(y) - u(x) \ : \ u \in \mathrm{Lip}(\Omega),\ \varphi^0(.,Du) \leq 1 \ a.e.\} \quad \leq \quad d^{\varphi}(x,y)$$

Proof. Let $u \in \mathrm{Lip}(\Omega)$ satisfying $\varphi^0(.,Du) \leq 1$ *a.e.*, and let N_u be the union of set of non differentiability of u and the set where $\varphi^0(.,Du) > 1$. For each $\gamma \in \mathrm{path}(x,y)$ and γ transversal to N_u we have

$$u(y) - u(x) \ = \ \int_0^1 (u \circ \gamma)'(t)dt \quad = \quad \int_0^1 \nabla u(\gamma(t)) \cdot \dot\gamma(t)dt.$$

We notice that for almost every $t \in]0,1[$ for which $\varphi(\gamma(t),\dot\gamma(t)) = 0$ one has $\nabla u(\gamma(t)) \cdot \dot\gamma(t) = 0$, otherwise $\varphi^0(\gamma(t), Du(\gamma(t))) = +\infty$ by the 1-homogeneity of φ. Moreover, for almost every $t \in]0,1[$ for which $\varphi(\gamma(t),\dot\gamma(t)) > 0$ one has

$$\nabla u(\gamma(t)) \cdot \dot\gamma(t) \leq \varphi(\gamma(t),\dot\gamma(t))$$

because of the 1-homogeneity of φ and $\varphi^0(.,Du(.)) \leq 1$.

We thus obtain

$$u(y) - u(x) \leq \int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt$$

and then

$$u(y) - u(x) \leq \inf\left\{\int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt \ : \ \gamma \in \mathrm{path}(x,y) \text{ and } \gamma \text{ transversal to } N_u\right\}.$$

The conclusion now follows using the definition of $d^{\varphi}(x,y)$. □

We finally prove that when the Finsler metric φ is regular, the two definitions above coincide.

Proposition A.2. *If $\varphi \in \mathcal{M}_\beta$ and $x \mapsto \varphi(x,q)$ is upper semicontinuous on Ω for all q, then $d^{\varphi} = d_{\varphi}$.*

Proof. The inequality $d^{\varphi} \geq d_{\varphi}$ is obvious, we just check the reverse inequality. To this end, we have to prove that for any $x, y \in \Omega$ and $N \in \mathcal{N}$ one has

$$d_{\varphi}(x,y) = \inf\left\{\int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt \ : \ \gamma \in \mathrm{path}(x,y) \text{ and } \gamma \text{ transversal to } N\right\}. \quad \text{(A.1)}$$

Let $\gamma \in \mathrm{path}(x,y)$, if γ is transversal to N then there is nothing to do. Otherwise, γ can be approximated strongly in $W^{1,\infty}(]0,1[,\Omega)$ by a sequence $(\gamma_k)_k$ in $\mathrm{path}(x,y)$ with γ_k transversal to N for all k (for example, see Lemma 3.2 in [5]). Then by the upper semicontinuity of $x \mapsto \varphi(x,q)$ we infer that

$$\limsup_{k\to+\infty} \int_0^1 \varphi(\gamma_k(t),\dot\gamma_k(t))dt \quad \leq \quad \int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt$$

so that

$$\int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt \geq \inf\left\{\int_0^1 \varphi(\gamma(t),\dot\gamma(t))dt \ : \ \gamma \in \mathrm{path}(x,y) \text{ and } \gamma \text{ transversal to } N\right\}.$$

and (A.1) holds, which concludes the proof. □

Appendix B. Properties of absolute minimizers, L and d_λ.

We first show that an absolute minimizer of (1.1) is indeed an optimal solution of (1.1).

Lemma B.1. *If the function $u \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ is an absolute minimizer of (1.1) then it is a minimizer of (1.1).*

Proof. Let $v \in g + W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$, we aim to show that

$$\operatorname*{ess.sup}_{x\in\Omega} H(x, Du(x)) \le \operatorname*{ess.sup}_{x\in\Omega} H(x, Dv(x)). \tag{B.1}$$

We first show that

$$\operatorname*{ess.sup}_{x\in\{u\neq v\}} H(x, Du(x)) \le \operatorname*{ess.sup}_{x\in\{u\neq v\}} H(x, Dv(x)). \tag{B.2}$$

To this end, we set $V_\delta^+ := \{x : u > v + \delta\}$ and $V_\delta^- := \{x : u < v - \delta\}$ for any positive δ. Then $V_\delta^+ \subset\subset \Omega$ and $v + \delta = u$ on ∂V_δ^+, and since u is an absolute minimizer of (1.1) we obtain

$$\operatorname*{ess.sup}_{x\in V_\delta^+} H(x, Du(x)) \le \operatorname*{ess.sup}_{x\in V_\delta^+} H(x, Dv(x)).$$

Passing to the limit as δ tends to zero yields

$$\operatorname*{ess.sup}_{x\in\{u>v\}} H(x, Du(x)) \le \operatorname*{ess.sup}_{x\in\{u>v\}} H(x, Dv(x))$$

and (B.2) follows by applying the same argument with V_δ^-.

We now conclude by noticing that $H(x, Du(x)) = H(x, Dv(x))$ for almost every x in $\{u = v\} \cap \Omega$. This and (B.2) conclude the proof of (B.1). □

We now turn to the study of L. We first notice that by definition and thanks to assumption (B), for any $\lambda \ge 0$ there exists $\beta \ge 0$ such that $L(.,.,\lambda)$ belongs to $\mathcal{M}_\beta$. In order to be in position to apply Proposition A.2, we prove the following regularity result on L.

Lemma B.2. *The function $(x, \lambda) \mapsto L(x, q, \lambda)$ is upper-semicontinuous on $\Omega \times \mathbb{R}_+$ for any $q \in \mathbb{R}^N$.*

Proof. Let $(x_n, \lambda_n)_n$ converge to $(x, \lambda) \in \Omega \times \mathbb{R}_+$, we must check that

$$\limsup_{n\to+\infty} L(x_n, q, \lambda_n) \le L(x, q, \lambda).$$

We may assume without loss of generality that the limsup is in fact a limit. For any $n \in \mathbb{N}$, we take $p_n \in \{H(x_n, .) \le \lambda_n\}$ such that $L(x_n, q, \lambda_n) = p_n \cdot q$. Thanks to (B), the sequence $(p_n)_n$ is bounded and we may extract a subsequence $(p_{n_k})_{n_k}$ converging to some $p \in \mathbb{R}^N$. Since H is l.s.c., we get $H(x, p) \le \lambda$, so that

$$\lim_{n\to+\infty} L(x_n, q, \lambda_n) = \lim_{k\to+\infty} p_{n_k} \cdot q_{n_k} = p \cdot q \le L(x, q, \lambda)$$

which concludes the proof. □

As a corollary of Propositions A.1 and A.2 and the above Lemma, we get the following.

Lemma B.3. *Let V be a connected open subset of Ω, then for any $\lambda \geq 0$ and $x, y \in V$ one has*

$$d_\lambda^V(x,y) \; = \; \sup\{u(y) - u(x) \; : \; u \in W^{1,\infty}(V) \cap \mathcal{C}(V), \; H(.,Du) \leq \lambda \text{ a.e. on } V\}. \quad \text{(B.3)}$$

Moreover, if V has Lipschitz boundary, then for any $\lambda \geq 0$ and $x, y \in \overline{V}$ one has

$$d_\lambda^V(x,y) \; = \; \sup\{u(y) - u(x) \; : \; u \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V}), \; H(.,Du) \leq \lambda \text{ a.e. on } V\}. \quad \text{(B.4)}$$

Proof. For (B.3), we first claim that

$$d_\lambda^V(x,y) \quad \geq \quad \sup\{u(y) - u(x) \; : \; u \in W^{1,\infty}(V) \cap \mathcal{C}(V), \; H(.,Du) \leq \lambda \text{ a.e. on } V\}.$$

We remark that by assumpions (A) and (C), the set $\{(L(z,\cdot,\lambda))^0 \leq 1\}$ coincides with the set $\{H(z,\cdot) \leq \lambda\}$ for any $z \in \Omega$. We infer from Lemma B.2 that $x \mapsto L(x,q,\lambda)$ is upper semicontinuous for any $q \in \mathbb{R}^n$, so that the claim follows by applying Propositions A.1 and A.2. The equality follows by taking $z \mapsto d_\lambda^V(x,z)$ as a test function in the sup (we recall that this function is admissible as a consequence of Remark 2.8 and Proposition 2.10).

When ∂V is Lipschitz regular, any function $u \in W^{1,\infty}(V) \cap \mathcal{C}(V)$ may be extended as a continuous function on $\overline{V}$, so that

$$d_\lambda^V(x,y) \quad = \quad \sup\{u(y) - u(x) \; : \; u \in W^{1,\infty}(V) \cap \mathcal{C}(\overline{V}), \; H(.,Du) \leq \lambda \text{ a.e. on } V\}$$

for any $x, y \in V$. We notice that the right hand side of this equality is continuous on $\overline{V} \times \overline{V}$ as a function of (x,y). Finally, since ∂V is Lipschitz regular then $(x,y) \mapsto d_\lambda^V(x,y)$ is also continuous on $\overline{V}$, so that (B.4) holds for $(x,y) \in \overline{V} \times \overline{V}$. □

In the course of section 4, we also need the two following technical results.

As a first consequence of assumption (D), we get that in a connected open set $V \subset\subset \Omega$, the pseudo-distances d_λ^V and d_λ^Ω locally coincide:

Lemma B.4. *Let $V \subset\subset \Omega$ be a connected open set, $\lambda^+ > \lambda^- > 0$, and assume that there exist and $M > \alpha > 0$ such that*

$$B(0,\alpha) \; \subset \; \{\; H(x,\cdot) < \lambda\} \; \subset \; B(0,M)$$

for all $x \in V$ and $\lambda \in [\lambda^-, \lambda^+]$. Then for any $x, y \in V$ such that $|y - x| < \frac{\alpha}{M} dist(x, \partial V)$ one has

$$\forall \lambda \in [\lambda^-, \lambda^+] \qquad \alpha|y-x| \; \leq \; d_\lambda^V(x,y) \; = \; d_\lambda^\Omega(x,y) \; \leq \; M|y-x|.$$

Proof. Let $\lambda \in [\lambda^-, \lambda^+]$. We first prove that

$$\alpha|y-x| \; \leq \; d_\lambda^V(x,y) \; \leq \; M|y-x|. \quad \text{(B.5)}$$

For the right left hand side inequality, we simply notice that for any $x \in V$ and $q \in \mathbb{R}^N$

$$\begin{aligned} L(x,q,\lambda) \quad &= \quad \sup\left\{p \cdot q \; : \; p \in \mathbb{R}^N, \, H(x,p) \leq \lambda\right\} \\ &\geq \quad \sup\left\{p \cdot q \; : \; p \in B(0,\alpha)\right\} \; \geq \; \alpha|q|, \end{aligned}$$

so that for any $\xi \in path_V(x,y)$ one has

$$\int_0^1 L(\xi(t), \dot{\xi}(t), \lambda) dt \geq \alpha \int_0^1 |\dot{\xi}(t)| dt \geq \alpha|y-x|$$

which concludes the proof. The proof of the right hand side inequality is analogue.

We now turn to the equality $d^V_\lambda(x,y) \;=\; d^\Omega_\lambda(x,y)$. It is clear that $d^V_\lambda(x,y) \;\geq\; d^\Omega_\lambda(x,y)$, and we infer from (B.5) that

$$d^\Omega_\lambda(x,y) \leq M|y-x| < \alpha\, dist(x,\partial V). \tag{B.6}$$

Let now $\xi \in path_\Omega(x,y)$, we notice that if for some $s\in]0,1[$ one has $|\xi(s)-x| \geq dist(x,\partial V)$ then

$$\begin{aligned}\int_0^1 L(\xi(t),\dot\xi(t),\lambda)dt \;&\geq\; \int_0^s L(\xi(t),\dot\xi(t),\lambda)dt\\ &\geq\; \alpha\int_0^s |\dot\xi(t)|dt \;\geq\; \alpha|\xi(s)-x| \;\geq\; \alpha dist(x,\partial V).\end{aligned}$$

It then follows from (B.6) that such a path is not optimal for d^Ω_λ, so that $d^V_\lambda(x,y) \;=\; d^\Omega_\lambda(x,y)$. This concludes the proof. □

Lemma B.5. *Let $V \subset\subset \Omega$ be a connected open set, $\lambda \geq 0$, $x,y\in V$ and $r>0$ such that $r < \min\{|y-x|, dist(x,\partial V), dist(y,\partial V)\}$. Then there exists $z\in V$ such that $|z-y| = r$ and*

$$d^V_\lambda(x,y) \;=\; d^V_\lambda(x,z) + d^V_\lambda(z,y).$$

Proof. Let $\varepsilon>0$ and $\xi\in path_V(x,y)$ be such that

$$d^V_\lambda(x,y) \;\geq\; \int_0^1 L(\xi(t),\dot\xi(t),\lambda)dt - \varepsilon.$$

By definition of r and continuity of ξ there exists some $s\in]0,1[$ such that $|\xi(s)-y| = r$. Then $z_\varepsilon := \xi(s)$ belongs to V and

$$\begin{aligned}d^V_\lambda(x,y) \;&\geq\; \int_0^s L(\xi(t),\dot\xi(t),\lambda)dt + \int_s^1 L(\xi(t),\dot\xi(t),\lambda)dt - \varepsilon\\ &\geq\; d^V_\lambda(x,z_\varepsilon) + d^V_\lambda(z_\varepsilon,y) - \varepsilon.\end{aligned}$$

The family $(z_\varepsilon)_{\varepsilon>0}$ is clearly bounded, let z be one of its cluster points as $\varepsilon\to 0$. Then $|z-y| = r$, so that $z\in V$, and taking the liminf in the above inequality yields

$$d^V_\lambda(x,y) \;\geq\; d^V_\lambda(x,z) + d^V_\lambda(z,y).$$

We conclude by applying the triangular inequality, which holds here since $z\in V$. □

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