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Γ-CONVERGENCE AND ABSOLUTE MINIMIZERS FOR SUPREMAL FUNCTIONALS

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Abstract. In this paper, we prove a semi-continuity theorem for supremal functionals whose supremand satisfy weak coercivity assumptions as well as a generalized Jensen inequality. The existence of minimizers for variational problems involving such functionals (together with a Dirichlet condition) easily follows from this result. We show the existence of at least one absolute minimizer (*i.e.* local solution) among these minimizers. We provide two different proofs of this fact relying on different assumptions and techniques.

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1. Introduction

The classical problems of Calculus of Variations are concerned with the minimization of integral functionals. From the point of view of modelization, this usually corresponds to minimizing the mean of a certain pointwise quantity which can be an energy density or a cost for unitary work. In other words, minimizing the functional is equivalent to controlling the mean of some quantity. On the other hand there are situations in which a control on the worst (or best) situation is needed, in the sense that one wants to control the maximum of certain quantity instead of its mean. This last type of problems can be easily formulated as a problem of Calculus of Variations or Optimal Control Theory in L^∞, and it received a lot of attention in the last ten years.

The problems of Calculus of Variations in L^∞ that we will study in this paper are formulated as follows:

$$\min_{v\in W^{1,\infty}_g(\Omega,\mathbb{R}^M)} \operatorname*{\mathit{esssup}}_{\Omega} f(x,v(x),Dv(x)), \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, f satisfies the natural assumption to have the measurability of $f(x,v(x),Dv(x))$ for all $v\in W^{1,\infty}(\Omega,\mathbb{R}^M)$, g is a map in $W^{1,\infty}(\Omega,\mathbb{R}^M)$ and $W^{1,\infty}_g(\Omega,\mathbb{R}^M)$ denotes the space of functions such that $(u-g)\in W^{1,\infty}(\Omega,\mathbb{R}^M)\cap\mathcal{C}_0(\Omega,\mathbb{R}^M)$.

Definition 1.1. According to part of the already existing litterature we will refer to functionals of the type

$$v\mapsto F(v):=\operatorname*{\mathit{esssup}}_{x\in\Omega} f(x,v(x),Dv(x)),$$

as *supremal* functionals.

Two relevant facts distinguish the classical integral problems of the Calculus of Variations from problems involving a supremal functional. The first one is that when one works with a supremal functional, sets of arbitrarily small measure can not be neglected. The second one is that the problem (1.1) is in some sense non-local. By this last sentence we mean that modifying a function v on a set of positive measure may not change the value of the functional $F(v)$. The first fact plays a role in studying the semicontinuity and relaxation of supremal functionals, while the second come into play in studying minimizers.

Both differences are due to the fact that integral functionals are additive while supremal functional are only subadditive with respect to the union of sets. The additivity property of integral functionals implies that a minimizer u for an integral functional G defined on some space of measurable functions X by:

$$G(v) = \int_\Omega g(x, v(x), Dv(x))dx$$

is always a local minimizer, which means that for all subdomain $V \subset \Omega$ one has

$$\int_V g(x, u(x), Du(x))dx \le \int_V g(x, v(x), Dv(x))dx$$

whenever $v \in X$ with $v = u$ on ∂V. By analogy with the case of integral functionals, Aronsson [2] defined the following notion of local minimizers for supremal functionals

Definition 1.2. A local minimizer for (1.1) is a minimizer u such that for all subdomain $V \subset \Omega$ one has

$$\mathop{esssup}_{x\in V} f(x, u(x), Du(x)) \le \mathop{esssup}_{x\in V} f(x, v(x), Dv(x))$$

for all v in $W^{1,\infty}$ such that $v = u$ on ∂V. In the following, a local minimizer will be called an AML.

The name AML stands for Absolute Minimizing Lipschitz and it was originally referred to the scalar case where $f(x, s, A) = |A|$. Before going further, we notice that a minimizer for a supremal functional is not necessarily a local minimizer.

Example 1.3. Consider the following variational problem on $\Omega :=]-2, 2[$:

$$\min\{ \mathop{esssup}_{x\in]-2,2[} f(x, v'(x)) \,:\, v(-2) = v(2) = 0\} \tag{1.2}$$

where the supremand f is given by

$$f(x, s) := \begin{cases} 1 + |s-1| & \text{if } x \in]-2, -1[\cup]1, 2[, \\ |s|/2 & \text{if } x \in [-1, 1]., \end{cases}$$

The set of optimal solutions of (1.2) is the set of Lipschitz continuous functions u such that: $u(x) = x + 2$ for $x \in [-2, -1]$, $u(x) = x - 2$ for $x \in [1, 2]$ and u has Lipschitz constant less that 2 on $[-1, 1]$. An AML of this problem should be an optimal solution of the localized problem on $V =]-1, 1[$, which then reads:

$$\min\{ \mathop{esssup}_{x\in]-1,1[} |v'(x)|/2 \,:\, v(-1) = 1, v(1) = -1 \textit{ and } v \textit{ solution of } (1.2)\}. \tag{1.3}$$

The optimal value of this problem is $1/2$, and it is attained only for the minimizer $\tilde{u}$ of (1.2) such that $\tilde{u}(x) = -x$ on $[-1, 1]$. As a consequence, no other minimizer is an AML (and it can also be shown that this function $\tilde{u}$ is indeed an AML of (1.2)).

Local minimizers are in some sense *better* than generic minimizers: when the supremand f is smooth enough, such minimizers are solution of an Euler equation, and in the good cases it can be proven that they satisfy a maximum principle (we refer to Barron *et al.* [8]). For example, the model problem

$$\min_{v\in W^{1,\infty}_g(\Omega)} \|Dv\|_{L^\infty(\Omega)} = \min_{v\in W^{1,\infty}_g(\Omega)} \underset{x\in\Omega}{esssup}|Dv(x)|, \tag{1.4}$$

admits a unique AML u which is the unique viscosity solution of $-\Delta_\infty(u) = 0$ in Ω.

The aim of this paper is twofold: we first prove a lower-semicontinuity theorem for supremal functionals (theorem 3.1) and then give two different existence theorems for AML (theorems 4.1 and 4.9). The lower-semicontinuity result improves some known results (*e.g.* Barron *et al.* [7]) and is based on the L^p approximation technique for supremal functionals and on a suitable rearrangement technique. With this result, the existence of minimizers for problems of the type (1.1) follows by the direct method of the Calculus of Variations. Then the problem of the existence of AML arises.

Our first existence theorem (th.4.1) for AMLs is based on the same L^p approximation and rearrangement arguments as in the proof of the lower-semicontinuity result. We notice that L^p approximations have been already widely used for this problem in the litterature (Aronsson [2], Barron *et al.* [8], Jensen [17] ...). Here we exploit the Γ-convergence character of this approximation (see proposition 3.6): this allows us to give a semicontinuity result under weak assumptions and also to distinguish which sequences of approximate minimizers converge to an AML and which *a priori* do not.

The second existence theorem (th.4.9) is based on a Perron-like method. It shows that the concept of AML is very natural for supremal functional and it gives an intrinsic proof (without approximation) for their existence. The method we use was introduced in [4] by Aronsson in the proof of the existence of an AML for the model problem (1.4), and it has been recently adapted to the metric setting by Juutinen [18].

The paper is organized as follows: section §2 contains preliminary results and definitions which shall be used through the paper. The lower semicontinuity result (theorem 3.1) is stated and proven in section §3; the L^p approximation is also introduced in this section (§3.2). Finally, the two existence theorems for AMLs are stated and proven in section §4.

2. Preliminary results

2.1. Γ-convergence.

Let X be a metric space, a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge to F at x if

$$\Gamma - \liminf F_n(x) = \Gamma - \limsup F_n(x), \tag{2.1}$$

where

$$\begin{cases} \Gamma - \liminf F_n(x) = \inf \left\{ \liminf F_n(x_n) : \quad x_n \to x \right\} \\ \Gamma - \limsup F_n(x) = \inf \left\{ \limsup F_n(x_n) : \quad x_n \to x \right\}. \end{cases} \tag{2.2}$$

The Γ−convergence was introduced by De Giorgi *et al.* in [15], for an introduction to this theory we refer to Dal Maso [13]. The following classical theorem reports the characterizing properties of the Γ-convergence.

Theorem 2.1. *Assume that $F_n \xrightarrow{\Gamma} F$ then F is lower semicontinuous on X. Moreover if F_n are equi-coercive on X then F is coercive too. In this last case, the following holds:*

(1) *the sequence $(\inf_X F_n)_n$ converges to the minimum of F on X,*
(2) *if x_n is such that $F_n(x_n) \leq \inf_X F_n + \varepsilon_n$, $\varepsilon_n \to 0$ and $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ then $F(x) = \min_X F$.*

2.2. A lemma on Young measures.

The Young measures are one of the basic tools of the Calculus of Variations. We will make use of this tool at some point. We report here some results we will need. The following proposition contains a fundamental property of Young measures (see for example Muller [19])

Proposition 2.2. *Suppose that a sequence of maps $(z_k)_k$ generate a Young measure ν. Let f be a Carathédory function and assume that the negative part $(f^-(.,z_k(.)))_k$ is weakly relatively compact in $L^1(E)$. Then*

$$\liminf_{k\to\infty} \int_E f(x, z_k(x))dx \geq \int_E \int_{\mathbb{R}^N} f(x,\lambda)d\nu_x(\lambda)dx.$$

Actually the assumption on the integrand f can be weakened and the previous proposition still holds if $f : E \times \mathbb{R}^N \to \mathbb{R}$ is assumed to be Borel measurable in both variables and lower semicontinuous in the second (see for example Berliocchi *et al.* [10]). We will use the following application of the previous proposition: let $f : \Omega \times R^M \times \mathbb{R}^{M\times N} \to \mathbb{R}$ be a non-negative function which is Borel measurable with respect to all variables and such that $f(x,\cdot,\cdot)$ is lower semicontinuous for almost every x in Ω. Suppose that $u_j \rightharpoonup u$ in $W^{1,p}(\Omega,\mathbb{R}^M)$ and that $(Du_j)_j$ generates the Young measures ν (such a Young measure is usually called a L^p-gradient Young measure, see Pedregal [20]). Then the couple (u_j, Du_j) generates the Young measure $x \to \delta_{u(x)} \otimes \nu_x$, and by the previous proposition and remark

$$\liminf_{j\to\infty} \int_\Omega f(x, u_j(x), Du_j(x))dx \geq \int_\Omega \int_{\mathbb{R}^{M\times N}} f(x, u(x), \lambda)d\nu_x(\lambda)dx.$$

2.3. level convex and Morrey quasiconvex functions.

We now introduce briefly the notions of level-convexity and Morrey-quasiconvexity which are strongly related to one of the main hypotheses of our lower semicontinuity result, namely that the supremand f satisfies a generalized Jensen inequality in the third variable (see (3.2) in theorem 3.1).

Definition 2.3. Let $f : \mathbb{R}^{M\times N} \to \overline{\mathbb{R}}$, f is a level convex function if all its sublevel sets are convex, that is if $E_\gamma = \{x \in \mathbb{R}^{M\times N} \ : \ f(x) \leq \gamma\}$ is convex for every $\gamma \in \mathbb{R}$.

The following result states that a generalized Jensen inequality holds for level convex functions.

Proposition 2.4. *Let $f : \mathbb{R}^{M\times N} \to \overline{\mathbb{R}}$ be lower semicontinuous and level convex. Let μ be a probability measure on $\mathbb{R}^{M\times N}$ and let $\phi \in L^1_\mu(\mathbb{R}^{M\times N}; \mathbb{R}^{M\times N})$ be a given function. Then*

$$f\left(\int_{\mathbb{R}^{M\times N}} \phi(A)d\mu(A)\right) \leq \mu - \underset{A\in\mathbb{R}^{M\times N}}{esssup}\, f(\phi(A)). \tag{2.3}$$

Proof. Define $\gamma = \mu - \underset{A\in\mathbb{R}^{M\times N}}{esssup}\, f(\phi(A))$ and $E_\gamma = \{A \in \mathbb{R}^{M\times N} \ : \ f(A) \le \gamma\}$, then for $\mu - a.e. A \in \mathbb{R}^{M\times N}$ one has $\phi(A) \in E_\gamma$. Since f is lower semicontinuous and level convex, E_γ is a closed convex set. Hence, since μ is a probability measure, $\int_{\mathbb{R}^{M\times N}} \phi d\mu$ belongs to E_γ, which concludes the proof. □

The second class of functions (introduced and studied in [7]) is the following:

Definition 2.5. $f : \mathbb{R}^{M\times N} \to \mathbb{R}$ is (strong) Morrey-quasiconvex if for any $\varepsilon > 0$, for any $A \in \mathbb{R}^{M\times N}$ and any $K > 0$ there exists a $\delta = \delta(\varepsilon, K, A) > 0$ such that if $\varphi \in W^{1,\infty}(Y, \mathbb{R}^M)$ satisfies

$$\|D\varphi\|_{L^\infty(Y)} \le K, \quad \max_{\partial Y} |\varphi(x)| \le \delta,$$

where $Y =]0,1[^N$, then

$$f(A) \le \underset{Y}{esssup} f(A + D\varphi).$$

It is not clear (and this was already pointed out in [7]) whether this last class of functions can be characterized in terms of some Jensen inequality. However thanks to the results of [7] this class of functions should be the natural setting for the semicontinuity of supremal functionals in the vectorial case $M > 1$.

3. A semicontinuity theorem

In this section we prove the following semicontinuity theorem:

Theorem 3.1. *Let $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}$ be a measurable function which is lower semicontinuous in the second and third variables. Assume that f is uniformly coercive in the third variable, i.e.*

$$\forall t \in \mathbb{R}, \quad \exists R, \quad \forall (x, s), \qquad \{A : f(x, s, A) \le t\} \subset B(0, R). \tag{3.1}$$

and that it satisfies the following generalized Jensen inequality

$$\forall (x, s), \qquad f\left(x, s, \int_{\mathbb{R}^{M\times N}} A d\nu_x(A)\right) \le \nu_x - \underset{A\in\mathbb{R}^{M\times N}}{esssup}\, f(x, s, A) \tag{3.2}$$

whenever $(\nu_x)_{x\in\Omega}$ is an L^p gradient Young measure for all $p \in (1,\infty)$. Then the functional

$$F(v) := \begin{cases} \underset{x\in\Omega}{esssup} f(x, v(x), Dv(x)) & \text{if } v \in W^{1,\infty}(\Omega; \mathbb{R}^M) \cap \mathcal{C}(\Omega; \mathbb{R}^M), \\ \\ +\infty & \text{elsewhere,} \end{cases} \tag{3.3}$$

is lower semicontinuous in $\mathcal{C}(\Omega; \mathbb{R}^M)$ for the uniform convergence of functions, and thus in $W^{1,\infty}(\Omega; \mathbb{R}^M)$ with respect to the w^* convergence.

Thanks to proposition 2.4, any level-convex function satisfies (3.2). Therefore, the above result applies to any level-convex supremand which satisfies the uniform coercivity condition. Notice in particular that convex functions are level-convex and then satisfy (3.2).

Remark 3.2. In the scalar case ($M = 1$), the hypothesis (3.2) characterizes the level-convexity, so that level-convexity seems to be the good assumption in this setting. In the vectorial case ($M > 1$), it is proven in [7] that under regularity assumptions on f stronger than that of theorem 3.1, (strong) Morrey quasiconvexity is a necessary condition for lower semicontinuity. It is still an open problem to investigate the relationship between inequality (3.2) and (strong) Morrey quasiconvexity.

Let's make a short comparison with the other semicontinuity theorems existing in litterature:

- In [7] the authors prove that the functional (3.3) is lower semicontinuous with respect to the $w^*-W^{1,\infty}$ convergence under the following assumptions on the supremand f:
  (1) f is (strong) Morrey quasiconvex (see definition 2.5) in the last variable,
  (2) there exists a function $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous in its first variable and non decreasing in the second such that
  $$|f(x_1, s_1, A) - f(x_2, s_2, A)| \le \omega(|x_1 - x_2| + |s_1 - s_2|, |A|).$$
  Notice that in this result, no coercivity assumption is made on the supremand f, whereas our coercivity assumption (3.1) is a cornerstone of our proof of theorem 3.1. However, the hypothesis (2) above rules out supremands such as
  $$f(x, A) := \begin{cases} \frac{1}{|x|}|A| & if\ x \ne 0, \\ 0 & if\ x = 0, \end{cases}$$
  to which our theorem applies (for $\Omega = B(0, 1)$ for example).
- In Gori *et al.* [16] the authors consider a Serrin-type semicontinuity theorem (i.e. without coercivity assumptions) in the scalar case for the functional (3.3). The main results states that if f is level-convex and for all $K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^N$ there exists a modulus of continuity ω_K such that
  $$|f(x_1, s_1, A) - f(x_2, s_2, A)| \le \omega_K(|x_1 - x_2| + |s_1 - s_2|),$$
  then (3.3) is lower semicontinuous with respect to the convergence defined by : $u_n \to u$ if and only if $u_n, u \in W^{1,\infty}$ and $u_n \to u$ uniformly.
- The semicontinuity for supremal functionals whose supremand do not depend on the gradient is studied in Acerbi *et al.* [1].

The rest of the section is devoted to the proof of theorem 3.1 which , although simple, will be divided in two steps. First, using a suitable rearrangement technique, we will simplify the settings reducing the condition of coercivity in the third variable to a condition of superlinearity. Second we will prove the semicontinuity in the superlinear case by approximation, using the fact that a Γ-limit is always semicontinuous.

3.1. **Towards a simplification.**

In this subsection, we reduce the proof of theorem 3.1 to the case where the supremand f is bounded from below and has linear growth at infinity (see proposition 3.6 below).

Lemma 3.3. *Assume that for any real number α, the functional F_α given by $F_\alpha(v) := \max\{F(v), \alpha\}$ is lower semicontinuous in $\mathcal{C}(\Omega; \mathbb{R}^M)$. Then the functional F is also l.s.c. in $\mathcal{C}(\Omega; \mathbb{R}^M)$.*

Proof. By contraddiction assume that there exist a sequence $(u_n)_n$ converging to u uniformly such that

$$\liminf_{n\to\infty} F(u_n) < F(u).$$

Then there exists $0 < \varepsilon$ and a subsequence still denoted by (u_n) such that $F(u_n) \leq F(u) - \varepsilon$ for n large enough. Let $\alpha := F(u) - \varepsilon$, then

$$\liminf_{n\to\infty} F_\alpha(u_n) \leq F(u) - \varepsilon < F(u) = F_\alpha(u),$$

which contradicts the lower-semicontinuity of F_α. □

It is thus sufficient to verify that F_α is lower semicontinuous for any $\alpha \in \mathbb{R}$. Notice that the functional F_α is associated to the supremand $\max\{f, \alpha\}$, and that this supremand obviously satisfies (3.1) and (3.2) whenever f does. We may thus assume that the supremand f is bounded from below by some real constant α. Since the functional associated to the supremand $f - \alpha$ is $F - \alpha$, we may assume that $\alpha = 0$. From now on, we will then assume that f is non-negative on $\Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N}$. As a consequence, the functional F may also be rewritten as follows

$$F(v) := \begin{cases} \|f(.,v(.),Dv(.))\|_{L^\infty(\Omega)} & \text{if } v \in W^{1,\infty}(\Omega;\mathbb{R}^M) \cap \mathcal{C}(\Omega;\mathbb{R}^M), \\ \\ +\infty & \text{elsewhere.} \end{cases} \tag{3.4}$$

The two following lemmas will help us reduce the proof of theorem 3.1 to the case where f has uniform linear growth in its third variable.

Lemma 3.4. *Let $h : \mathbb{R} \to \mathbb{R}_+$ be non decreasing and lower semicontinuous, then for all measurable function g on Ω, one has*

$$\operatorname*{\mathit{esssup}}_{x\in\Omega} (h \circ g(x)) \quad = \quad h(\operatorname*{\mathit{esssup}}_{x\in\Omega} g(x)).$$

The proof is left to the reader.

Lemma 3.5. *Suppose that $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}_+$ satisfies the uniform coercivity condition (3.1). Then there exists an increasing continuous function $h : [-1,+\infty) \to \mathbb{R}_+$ such that for all (x,s,A) one has $(h \circ f)(x,s,A) \geq \|A\|$.*

Proof. For any $r \geq 0$, we set

$$H(r) := \inf\left\{f(x,s,A) \ : \ x \in \Omega, s \in \mathbb{R}^M, \|A\| \geq r\right\}.$$

Then for all (x,s,A) one has $f(x,s,A) \geq H(\|A\|)$. Moreover, we notice that H is nondecreasing and that $H(r) \to +\infty$ as r goes to infinity since f is uniformly coercive. Therefore there exists an increasing sequence $(r_n)_{n\in\mathbb{N}}$ of real numbers such that: $r_0 = 0$, and $r_n \geq r_{n-1} + 1$ as well as $H(r_n) \geq H(r_{n-1}) + 1$ for any $n \geq 1$. We define H^* to be the only piecewise affine function on $\mathbb{R}_+$ with value $k + n - 1$ at each point r_n with $n \geq 1$ and with value -1 at $r_0 = 0$. Notice that H^* is increasing, continuous, one-to-one from $\mathbb{R}_+$ to $[-1,\infty)$ and that $H \geq H^*$. Let $h : [-1,\infty) \mapsto \mathbb{R}_+$ be its inverse, then h has the desired properties since

$$\forall (x,s,A) \qquad (h \circ f)(x,s,A) \geq (h \circ H)(\|A\|) \geq (h \circ H^*)(\|A\|) = \|A\|.$$

□

It follows fron lemma 3.4 that if h is an increasing and continuous function defined on $\mathbb{R}_+$, then the functional associated to the supremand $h \circ f$ is simply $h \circ F$ (where we of course set $h(+\infty) = +\infty$). Obviously, $h \circ F$ is l.s.c. on $\mathcal{C}(\Omega; \mathbb{R}^M)$ if and only if F is so. Since h is increasing, it is also clear that $h \circ f$ satisfies (3.1) and (3.2) whenever f does.

3.2. **Semicontinuity by approximation.**
As a consequence of the above lemmas and comments, the proof of theorem 3.1 now reduces to that of the following proposition.

Proposition 3.6. *Let $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times N} \to \mathbb{R}_+$ be a measurable function which is lower semicontinuous in the second and third variable. Assume that f satisfies the generalized Jensen inequality (3.2), and that it has linear growth in its third variable, i.e. there exists a positive constant c such that*

$$\forall (x, s, A) \qquad f(x, s, A) \geq c \|A\|. \tag{3.5}$$

Then the functional F given by (3.3) is lower semicontinuous in $\mathcal{C}(\Omega; \mathbb{R}^M)$ and then in $W^{1,\infty}(\Omega; \mathbb{R}^M)$ with respect to the w^*-convergence.

Proposition 3.6 will be an easy consequence of proposition 3.7 below. The proof is now rather classical (see Barron *et al.* [8], Bhattacharya *et al.* [9]) although we make here a sharp use of the generalized Jensen inequality (3.2).

Proposition 3.7. *Let $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M \times N} \to \mathbb{R}_+$ be a measurable function which is lower semicontinuous in the second and third variable. Assume that f satisfies the generalized Jensen inequality (3.2), and that it has linear growth in its third variable. For any $p > N$, we define the functional $F_p : \mathcal{C}(\Omega; \mathbb{R}^M) \to \mathbb{R}_+$ by*

$$F_p(v) = \begin{cases} \left(\displaystyle\int_\Omega f(x, v(x), Dv(x))^p dx \right)^{1/p} & \text{if } v \in W^{1,p}(\Omega; \mathbb{R}^M) \cap \mathcal{C}(\Omega; \mathbb{R}^M), \\ +\infty & \text{otherwise.} \end{cases}$$

Then the family $(F_p)_{p>N}$ $\Gamma-$converges to F in $\mathcal{C}(\Omega; \mathbb{R}^M)$ as p goes to $+\infty$.

Proof. We first notice that for any v in $\mathcal{C}(\Omega; \mathbb{R}^M)$, one has

$$\limsup_{p \to \infty} \; F_p(v) \quad \leq \quad F(v). \tag{3.6}$$

Indeed, if $F(v) = +\infty$, there is nothing to prove, and if $F(v) < +\infty$, then $f(\cdot, v(\cdot), Dv(\cdot))$ belongs to $L^p(\Omega)$ for every $p \geq 1$ and

$$\lim_{p \to \infty} \|f(\cdot, v(\cdot), Dv(\cdot))\|_{L^p(\Omega)} = ||f(\cdot, v(\cdot), Dv(\cdot))||_{L^\infty(\Omega)},$$

so that (3.6) follows from (3.4). As a consequence, $\Gamma - \limsup F_p \leq F$.

It remains to prove that $\Gamma - \liminf F_p \geq F$. Let $(v_p)_{p>N}$ be a family in $\mathcal{C}(\Omega; \mathbb{R}^M)$ converging uniformly on Ω to some function v, we must check that

$$\liminf_{p \to \infty} \; F_p(v_p) \quad \geq \quad F(v). \tag{3.7}$$

We may assume without loss of generality that there exists C in $\mathbb{R}$ such that $F_p(v_p) \leq C$ for any $p > N$. Then for any real numbers p, q with $p > q > N$, one has

$$\begin{aligned}\left(\int_\Omega |Dv_p(x)|^q dx\right)^{1/q} &\leq \frac{1}{c}\left(\int_\Omega f(x,v_p(x),Dv_p(x))^q dx\right)^{1/q}\\ &\leq \frac{1}{c}\left[|\Omega|^{1-q/p}F_p(v_p)^q\right]^{1/q} \leq \frac{C}{c}|\Omega|^{1/q-1/p}. \qquad (3.8)\end{aligned}$$

Therefore, the family $(Dv_p)_{p>q}$ is bounded in $L^q(\Omega;\mathbb{R}^{N\times M})$ for all $q>N$, so that the family $(v_p)_p$ converges to v weakly in $W^{1,q}$ for any such real number q. Thus the family $(Dv_p)_{p>N}$ generates a Young measure $\{\nu_x\}_{x\in\Omega}$ such that

$$Dv(x) = \int_{\mathbb{R}^{M\times N}} A d\nu_x(A)$$

for almost every x in Ω. We infer from the discussion following proposition 2.2 that for any fixed $q>r>N$ one has

$$\begin{aligned}\liminf_{p\to\infty} F_q(v_p) &= \liminf_{p\to\infty}\left(\int_\Omega f(x,v_p(x),Dv_p(x))^q dx\right)^{1/q}\\ &\geq \left[\int_\Omega\int_{\mathbb{R}^{M\times N}} f(x,v(x),A)^q d\nu_x(A)dx\right]^{1/q}\\ &\geq \left[\int_\Omega\left(\int_{\mathbb{R}^{M\times N}} f(x,v(x),A)^r d\nu_x(A)\right)^{q/r} dx\right]^{1/q},\end{aligned}$$

where the last inequality follows from the convexity of $t\mapsto t^{q/r}$ on $\mathbb{R}_+$. Passing to the liminf in q, we get

$$\liminf_{q\to\infty}\liminf_{p\to\infty} F_q(v_p) \geq \mathop{esssup}_{x\in\Omega}\left[\left(\int_{\mathbb{R}^{M\times N}} f(x,v(x),A)^r d\nu_x(A)\right)^{1/r}\right].$$

Letting now r tend to $+\infty$ then yields

$$\liminf_{q\to\infty}\liminf_{p\to\infty} F_q(v_p) \geq \mathop{esssup}_{x\in\Omega}\left[\nu_x - \mathop{esssup}_{A\in\mathbb{R}^{M\times N}} f(x,v(x),A)\right].$$

We now apply the extended Jensen inequality (3.2) with the gradient Young measures $(\nu_x)_{x\in\Omega}$ to obtain

$$f(x,v(x),Dv(x)) = f\left(x,v(x),\int_{\mathbb{R}^{M\times N}} A d\nu_x\right) \leq \nu_x - \mathop{esssup}_{A\in\mathbb{R}^{M\times N}} f(x,v(x),A)$$

for almost every x in Ω. Hence we infer

$$\begin{aligned}\mathop{esssup}_{\Omega} f(x,v(x),Dv(x)) &\leq \mathop{esssup}_{\Omega}\left[\nu_x - \mathop{esssup}_{A\in\mathbb{R}^{M\times N}} f(x,v(x),A)\right]\\ &= \liminf_{q\to\infty}\liminf_{p\to\infty} F_q(v_p)\\ &\leq \liminf_{q\to\infty}\liminf_{p\to\infty} |\Omega|^{1/q-1/p}F_p(v_p)\end{aligned}$$

which clearly yields $F(v)\leq\liminf F_p(v_p)$. □

Remark 3.8. In particular it follows from the inequality (3.8) and the semicontinuity of the L^q-norms that $F_p(v_p)\leq C$ and $v_p\to v$ uniformly implies that $v\in W^{1,\infty}$.

Remark 3.9 (Fixing Boundary data)**.** Notice that the previous result also holds if we add boundary data to the functionals F_p defining: $F_p : \mathcal{C}(\Omega;\mathbb{R}^M) \to \mathbb{R}_+$ as follows

$$F_p(v) = \begin{cases} \displaystyle\int_\Omega f(x, v(x), Dv(x))^p dx)^{1/p} & if\ u \in W^{1,p}_g(\Omega;\mathbb{R}^M) \cap \mathcal{C}(\Omega;\mathbb{R}^M), \\ +\infty & otherwise. \end{cases}$$

In fact the sequence chosen for the $\Gamma - \limsup$ inequality is trivial while the $\Gamma - \liminf$ inequality does not depend on the boundary data.

We conclude with the proof of proposition 3.6.

Proof. Since the family $(F_p)_{p>N}$ $\Gamma-$converges to F in $\mathcal{C}(\Omega;\mathbb{R}^M)$ as p goes to $+\infty$ and since Γ-limits are always lower semicontinuous (*e.g.* proposition 6.8 in [13]), the functional F is lower semicontinuous on $\mathcal{C}(\Omega;\mathbb{R}^M)$. □

4. Existence of local minimizers

We now turn to the existence of local minimizers for the variational problem $P(g,\Omega)$ given by

$$P(g,\Omega) \qquad Inf\left\{F(v,\Omega) \,:\, v \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\overline{\Omega}),\ v = g\ on\ \partial\Omega\right\}$$

where g is a Lipschitz function defined on $\partial\Omega$ and F is the supremal functional given by

$$F(v,V) := \begin{cases} \underset{x\in V}{esssup} f(x, v(x), Dv(x)) & \text{if } v \in W^{1,\infty}(V) \cap \mathcal{C}(V), \\ +\infty & \text{if } v \in \mathcal{C}(V) \setminus W^{1,\infty}(V), \end{cases}$$

where V is an open subset of Ω. Notice that in this last part, we restrict ourselves to the scalar case $M = 1$. Under the assumptions of theorem 3.1, the functional F satisfies the following property:

(H_0) $\quad F(.,\Omega)$ is l.s.c. and coercive on $\mathcal{C}_g(\overline{\Omega}) := \{v \in \mathcal{C}(\overline{\Omega}) \,:\, v = g\ \ on\ \ \partial\Omega\}$.

In (H_0), coercive means that the sublevel sets $\{F \leq t\}$ (for t in $\mathbb{R}$) are relatively compact in $\mathcal{C}_g(\overline{\Omega})$ for the topology of the uniform convergence. Under hypothesis (H_0), problem $P(g,\Omega)$ admits at least one solution. We shall denote by $S(P(g,\Omega))$ the set of optimal solutions to $P(g,\Omega)$. With these notations, a function u is an AML of $P(g,\Omega)$ if and only if u belongs to $S(P(g,\Omega))$ and to $S(P(u,V))$ for any open subset V of Ω. In this section, we show that, under mild assumptions on the supremand f or on the supremal funtional F, at least one of these solutions is a local solution of $P(g,\Omega)$. This result will be achieved in two different ways, and with two different sets of hypotheses.

4.1. **Existence by approximation.**

In this section, we use the L^p approximation method to prove the existence of an AML. The proof mainly relies on proposition 4.4 which is very similar to that of lemma 2.4 in [8] and is given here for the sake of completeness. For this method to work, some continuity assumption on the supremand f is necessary in addition to the hypotheses of theorem 3.1.

Theorem 4.1. *Let $f : \Omega \times \mathbb{R}^M \times \mathbb{R}^{M\times N} \to \mathbb{R}$ be a measurable function which is continuous in the second variable and lower semicontinuous in the third. Assume that f is uniformly coercive in the third variable (see (3.1)), that it satisfies the generalized Jensen inequality (3.2) and that f is bounded from below by a constant α, then problem $P(g,\Omega)$ admits at least one AML.*

The same rearrangement technique used for the semicontinuity allows to reduce the problem to the case in which f is non-negative and has linear growth at infinity. The approximating functionals we consider are that introduced in remark 3.9 and given by

$$F_p(v,V) = \begin{cases} \left(\int_\Omega f(x,v(x),Dv(x))^p dx\right)^{1/p} & if\, v \in W^{1,p}(V)\cap \mathcal{C}(V), \\ +\infty & if\, v \in \mathcal{C}(V)\setminus W^{1,p}(V), \end{cases}$$

where V is an open subset of Ω. In order to have the convergence of minima and minimizers we need the following

Lemma 4.2 (Equicoercivity)**.** *The family of functionals $(F_p(.,\Omega))_{p\geq N+1}$ is equicoercive in $\mathcal{C}_g(\overline{\Omega})$ with respect to the uniform convergence of functions.*

Proof. It is a consequence of inequality (3.8) and of the compact imbeddings of Sobolev spaces. □

Then thanks to theorem 2.1 the following convergence of the approximate minimizers holds:

Proposition 4.3. *Let ε_p be a sequence of positive numbers such that $\varepsilon_p \to 0$ and let u_p be a sequence of functions such that $F_p(u_p,\Omega) \leq \inf F_p(.,\Omega) + \varepsilon_p$. Let $(u_{p_i})_i$ be a subsequence of $(u_p)_p$ converging uniformly on Ω to some function u_∞. Then u_∞ is a minimizer for problem $P(g,\Omega)$ (in particular $u_\infty \in W^{1,\infty}$).*

Proof. This follows from proposition 3.7, remark 3.9, lemma 4.2 and theorem 2.1. □

The problem is now to understand if the minimizers obtained in proposition 4.3 are AML or not. In the next theorem and in the following example we will see that a quantitative argument is needed in order to guarantee that a sequence of approximate minimizers tends to an AML. This is due to the fact that the functionals we are considering are only subadditive with respect to the union of sets instead of being additive as the usual functionals of the Calculus of Variations.

Proposition 4.4. *Let $(u_p)_p$ be a sequence of functions such that $F_p^p(u_p,\Omega) \leq \inf F_p^p(.,\Omega) + \varepsilon_p$ where the sequence $(\varepsilon_p)_p$ is chosen so that $\lim_{p\to\infty} \sqrt[p]{\varepsilon_p} = 0$. Let $(u_{p_i})_i$ be a subsequence of $(u_p)_p$ converging uniformly on Ω to some function u_∞. Then u_∞ is an AML for problem $P(g,\Omega)$.*

Proof. From proposition 4.3 we already know that u_∞ is a minimizer of problem $P(g,\Omega)$. Let $V \subset \Omega$ be open and let $\varphi \in W^{1,\infty}(\Omega)$ such that we have $u_\infty + \varphi = u_\infty$ on ∂V.

We first notice that we can assume that $\varphi > 0$ in V. Indeed, if this is not the case

$$\begin{aligned} F(u_\infty+\varphi,V) = \max\{&F(u_\infty+\varphi,\{\varphi>0\}), \\ &F(u_\infty+\varphi,\{\varphi<0\}), F(u_\infty+\varphi,\{\varphi=0\})\}, \end{aligned} \tag{4.1}$$

and then our next arguments will apply to the first two elements of the set on the right side of (4.1) while the third does not really matter in the comparison with $F(u_\infty, V)$. Fix $\delta > 0$ such that $|\{x \in V : \varphi > \delta\}| > 0$. Since $(u_p)_p$ converges uniformly to u_∞, there exists p_δ such that for every $p \geq p_\delta$ we have $u_p + \frac{\delta}{2} > u_\infty > u_p - \frac{\delta}{2}$ in Ω. In particular $V_\delta := \{x \in V : \varphi > \delta\}$ is included in $V_{p,\delta} := \{x \in V : u_p + \frac{\delta}{2} < u_\infty + \varphi\}$ for such real numbers p. Notice that $V_{p,\delta}$ is an open set and that on its boundary one has $u_p = -\frac{\delta}{2} + u_\infty + \varphi$. Thus, for every $p \geq p_\delta$, we have

$$F_p^p(u_p, V_{p,\delta}) \leq F_p^p(u_\infty + \varphi - \frac{\delta}{2}, V_{p,\delta}) + \varepsilon_p$$

and then

$$F_p(u_p, V_\delta) \leq F_p(u_\infty + \varphi - \frac{\delta}{2}, V_{p,\delta}) + \sqrt[p]{\epsilon_p} \leq F(u_\infty + \varphi - \frac{\delta}{2}, V)|V|^{1/p} + \sqrt[p]{\epsilon_p}.$$

By remark 3.9,

$$F(u_\infty, V_\delta) \leq \liminf_{p \to +\infty} F_p(u_p, V_\delta) \leq F(u_\infty + \varphi - \frac{\delta}{2}, V).$$

Passing to the limit for $\delta \to 0^+$ and using the continuity of f with respect to the second variable, we obtain

$$F(u_\infty, V) \leq F(u_\infty + \varphi, V)$$

which concludes the proof. □

The next example shows that the control on the behaviour of ε_p when $p \to 0$ is necessary.

Example 4.5. Let v_∞ be a minimizer of F which is not an AML. Then as $F_p(v_\infty) \to F(v_\infty)$ the sequence $v_p = v_\infty$ for all p is a sequence of approximate minimizers, indeed if we denote by M_p the infimum of F_p and by M the infimum of F we have that

$$|F_p(v_\infty) - M_p| \leq |F_p(v_\infty) - M| + |M - M_p|$$

and the right hand term tends to 0. On the other hand it is clear that the sequence converges to v_∞ which is a minimizer but not an AML by assumption.

Corollary 4.6. *If for any p the functional F_p admits a least one minimizer and if those minimizers converge (up to subsequences) as $p \to \infty$ to some function u, then u is an AML.*

Remark 4.7. The need to specify the behavious of $\varepsilon_p \to 0$ in theorem 4.4 reflects the instability when $p \to \infty$ of the so called *fundamental estimates* which are the foundamental tools to prove the convergence of local minimizers (see Dal Maso *et al.* [14]).

4.2. **An intrinsic approach to existence of AML.**

Here we show the existence of an AML without making use of the approximation process of the last section, but by applying a Perron-like method. This proof relies on rather natural hypotheses on the functional F and on a connectedness assumption on the optimal set of problem $P(g, \Omega)$. Let us detail the assumptions of theorem 4.9 below. We first need to make more precise hypothesis (H_0) and set

(H_1) $F(., V)$ is l.s.c. and coercive on $\mathcal{C}_w(\overline{V}) := \{v \in \mathcal{C}(\overline{V}) \ : \ v = w \ \ on \ \ \partial V\}$ for any open subset V of Ω and w Lipschitz on ∂V.

Of course F satisfies (H_1) as soon as the hypotheses of theorem 3.1 are fullfilled. We shall also need the following additional hypothesis on the structure of the optimal sets of $S(P(g,\Omega))$:

(H_2) for any open subset V of Ω, w Lipschitz on ∂V and $y \in V$, the image set $\{u(y) : u \in S(P(w,V))\}$ is connected.

The hypothesis (H_2) is automatically satisfied if the functional F is for example convex, or level-convex in the variable v. The following lemma shows that it is also the case if for example the supremand $(x,s,A) \mapsto f(x,s,A)$ does not depend on the variable s.

Lemma 4.8. *Assume that the supremal functional F is given by*

$$F(v,V) := \begin{cases} \underset{x\in V}{esssup} f(x,Dv(x)) & \text{if } v \in W^{1,\infty}(V) \cap \mathcal{C}(V), \\ +\infty & \text{elsewhere,} \end{cases}$$

Then F satisfies hypothesis (H_2).

Proof. Let V be an open subset of Ω, w be a Lipschitz function on ∂V and y belong to V. Assume that u and v are two solutions of $P(w,V)$ and that $\alpha \in]u(y),v(y)[$. We then set $\tilde{u} := \max\{u, v+\alpha-v(y)\}$: clearly one has $\tilde{u}(y) = \alpha$ and $\tilde{u} = w$ on ∂V. Moreover, we deduce from lemma 4.10 below that $\tilde{u}$ is a solution of $P(w,V)$. Thus α belongs to $\{u(y) : u \in S(P(w,V))\}$. □

The main result of this section thus reads:

Theorem 4.9. *Assume that the functional F is such that (H_1) and (H_2) hold. Then the problem $P(g,\Omega)$ admits at least one AML.*

The rest of this section is devoted to the proof of theorem 4.9.

Lemma 4.10. *If u and v are two solutions of $P(g,\Omega)$, then $\max(u,v)$ and $\min(u,v)$ are also solutions of $P(g,\Omega)$.*

Proof. It is sufficient to notice that if V denotes the open set where $u > v$, then

$$\begin{aligned} F(\max(u,v),\Omega) &= \max\left(\underset{V}{esssup} f(x,u(x),Du(x)); \underset{\Omega\setminus V}{esssup} f(x,v(x),Dv(x)) \right) \\ &\leq \max\left(F(u,\Omega),F(v,\Omega)\right) = Inf(P(g,\Omega)) \end{aligned}$$

so that $\max(u,v)$ belongs to $S(P(g,\Omega))$. The proof is analogous for $\min(u,v)$. □

Proposition 4.11. *Assume that (H_1) holds. The functions $S^-(g,\Omega) : x \mapsto \inf_{u\in S(P(g,\Omega))} u(x)$ and $S^+(g,\Omega) : x \mapsto \sup_{u\in S(P(g,\Omega))} u(x)$ are solutions of $P(g,\Omega)$.*

Proof. We only prove that $S^+(g,\Omega)$ belongs to $S(P(g,\Omega))$. Let $(x_n)_{n\in\mathbb{N}}$ be a denumerable family which is dense in Ω. For any integers n and $i \leq n$, there exists a solution u_n^i of $P(g,\Omega)$ such that

$$u_n^i(x_i) \geq S^+(g,\Omega)(x_i) - \frac{1}{n}.$$

Then lemma 4.10 implies that the function $u_n := \max(u_n^1,\ldots,u_n^n)$ is a solution of $P(g,\Omega)$. By definition, it is such that

$$\forall i \leq n \qquad u_n \geq S^+(g,\Omega)(x_i) - \frac{1}{n}. \tag{4.2}$$

Since the sequence $(u_n)_n$ is a sequence of solutions of $P(g,\Omega)$, it is equilipschitzian. As the sequence $(x_n)_{n\in\mathbb{N}}$ is dense in Ω and $u_n \leq S^+(g,\Omega)$ on Ω, we deduce from (4.2) that $(u_n)_n$ converges uniformly on $\overline{\Omega}$ to $S^+(g,\Omega)$. Since F is l.s.c. on $\mathcal{C}(\overline{\Omega})$, we conclude that $S^+(g,\Omega)$ is a solution of $P(g,\Omega)$. □

Definition 4.12. We say that $u \in \mathcal{C}(\overline{\Omega})$ is a **local subsolution** of $P(g,\Omega)$ if u is a solution of $P(g,\Omega)$ and for all open subset V of Ω, $u \leq S^+(u,V)$ on V.

Analogously, $u \in \mathcal{C}(\overline{\Omega})$ is a **local supersolution** of $P(g,\Omega)$ if u is a solution of $P(g,\Omega)$ and for all open subset V of Ω, $u \geq S^-(u,V)$ on V.

We shall denote $S^{+loc}(P(g,\Omega))$ (resp. $S^{-loc}(P(g,\Omega))$) the set of all local supersolutions (resp. subsolutions) of $P(g,\Omega)$.

Lemma 4.13. *Assume that (H_1) holds. The function $S^+(g,\Omega)$ is a local supersolution of $P(g,\Omega)$, and $S^-(g,\Omega)$ is a local subsolution of $P(g,\Omega)$.*

Proof. It is sufficient to verify that $S^+(g,\Omega)$ is a local supersolution of $P(g,\Omega)$. Let V be an open subset of Ω, we define a function u on Ω by

$$u := \begin{cases} max\{S^-(S^+(g,\Omega),V); S^+(g,\Omega)\} & on \ \ V \\ S^+(g,\Omega) & on \ \ \Omega\setminus V \end{cases}$$

Since $F(S^-(S^+(g,\Omega),V),V)) \leq F(S^+(g,\Omega),V)$ and $u = S^+(g,\Omega)$ on $\Omega\setminus V$, u is a solution of $P(g,\Omega)$. By definition, u is thus lower than $S^+(g,\Omega)$ on Ω, so that $S^+(g,\Omega) \geq S^-(S^+(g,\Omega),V)$ on V. □

Remark 4.14. The same argument as in the proof of the above proposition allows to show a stronger property of the solution $S^+(g,\Omega)$: in fact, for every open subset V of Ω, $S^+(g,\Omega)$ is greater or equal than $S^+(S^+(g,\Omega),V)$.

Proposition 4.15. *Assume that (H_1) and (H_2) hold. A function u is an AML of $P(g,\Omega)$ if and only if it is both a local supersolution and a local subsolution of $P(g,\Omega)$.*

Proof. We first notice that if u is an AML of $P(g,\Omega)$, then for every open subset V of Ω, u is a solution of $P(u,V)$ so by definition, $S^-(u,V) \leq u \leq S^+(u,V)$. As a consequence, u is both a local supersolution and a local subsolution of $P(g,\Omega)$, which concludes the proof of the *if* part.

Now suppose that a function u is both a local supersolution and a local subsolution of $P(g,\Omega)$. Then in particular it is a solution of $P(g,\Omega)$.

Let now V be an open subset of Ω, we must check that u is a solution of $P(u,V)$. Let $(x_n)_{n\in\mathbb{N}}$ be a denumerable family which is dense in V. We construct by induction on n a family $(u_n)_{n\in\mathbb{N}}$ of solutions of $P(u,V)$ such that for any integers n and $i \leq n$ one has $u_n(x_i) = u(x_i)$. To this end, we notice that since u is both a local supersolution and a local subsolution of $P(g,\Omega)$, we have $S^-(u,V) \leq u \leq S^+(u,V)$ on V. Then thanks to hypothesis (H_2), there exists a solution u_1 of $(P(u,V))$ such that $u_1(x_1) = u(x_1)$. Now we assume that u_n has been constructed and we define u_{n+1}. To this end, we consider the problem $P(u,V_n)$ where $V_n := V\setminus\{x_1,\dots,x_n\}$: since the function u_n is a solution of $P(u,V)$ which is equal to u on the set $\{x_1,\dots,x_n\}$, we infer that it is an admissible function for $P(u,V_n)$. As a consequence, the value of the infimum of the problem $P(u,V_n)$ is equal to that of $P(u,V)$, and the optimal set $S(P(u,V_n))$ is included in $S(P(u,V))$. Once again, since u is both a local supersolution and a local subsolution of $P(g,\Omega)$, we have $S^-(u,V_n) \leq u \leq S^+(u,V_n)$ on V_n. Therefore, (H) yields the existence of a solution u_{n+1} of $(P(u,V_n))$ which is equal to u at the point x_{n+1}. This function u_{n+1} has the desired properties: it is a solution of $(P(u, V_n))$ and thus of $(P(u, V))$, it is equal to u at x_{n+1}, and since it is admissible for $P(u, V_n)$, it is also equal to u on $\{x_1, \dots, x_n\}$. As a family of solutions of $P(u, V)$, the family $(u_n)_{n\in\mathbb{N}}$ is equilipschitzian on V, and since $(x_n)_{n\in\mathbb{N}}$ is dense in V, it converges uniformly to u on $\overline{V}$. We then infer from (H_1) that u is a solution of $P(u, V)$, which concludes the *only if* part of the proof. □

Proposition 4.16. *Assume that (H_1) holds. The function $U^+ : x \mapsto \sup\{u(x) : u \in S^{-loc}(P(g, \Omega))\}$ is both a local supersolution and a local subsolution of $P(g, \Omega)$.*

Proof. The same argument as in the proof of proposition 4.11 shows that the function U^+ may be obtained as a uniform limit of local subsolutions of $P(g, \Omega)$, so it is a solution of $P(g, \Omega)$.

Let us check that U^+ is a local subsolution of $P(g, \Omega)$. Let V be an open subset of Ω, and assume that for some x_0 in V, one has $U^+(x_0) > S^+(U^+, V)(x_0)$. Then by definition of U^+, there exists a local subsolution u of $P(g, \Omega)$ such that $u(x_0) > S^+(U^+, V)(x_0)$. Since u is lower than U^+, it is also lower than $S^+(U^+, V)$ on ∂V. Let us set $A := \{x : u(x) > S^+(U^+, V)(x)\}$, then we must have $u \le S^+(u, A) = S^+(S^+(U^+, V), A)$ on A since u is a local subsolution of $P(g, \Omega)$. But the same argument as in the proof of lemma 4.13 yields that $S^+(S^+(U^+, V), A)$ is lower than $S^+(U^+, V)$ on A. As a consequence, we have $u \le S^+(U^+, V)$ on A, which is a contradiction.

Let us check that U^+ is a local supersolution of $P(g, \Omega)$. Let V be an open subset of Ω and $A = \{x : U^+(x) < S^-(U^+, V)(x)\}$. We define the function u on Ω by

$$u := \begin{cases} S^-(U^+, V) & \textit{on } A \\ U^+ & \textit{on } \Omega \setminus \overline{A}. \end{cases}$$

We claim that u is a local subsolution of $P(g, \Omega)$. Indeed, let B be an open subset of Ω, we must check that u is lower than $S^+(u, B)$ on B. By contradiction, assume that the set $C := \{x \in B : u(x) > S^+(u, B)(x)\}$ is not empty. We then claim that the set $D := \{x \in B : U^+(x) > S^+(u, B)(x)\}$ is also not empty: otherwise, C is included in A, but $u = S^-(U^+, V)$ on A and proposition 4.13 yields that $S^-(U^+, V)$ is a local subsolution of $(P(U^+, V))$ so that one should have $S^-(U^+, V) \le S^+(S^-(U^+, V), C) = S^+(u, C)$ on C: since remark 4.14 implies that $S^+(u, B) \ge S^+(u, C)$ on C, we get a contradiction. Thus the set D is not empty, but since $u = S^+(u, B)$ on ∂B and $u \ge U^+$ on Ω (and thus on ∂B), one has $U^+ = S^+(u, B)$ on ∂D. Since U^+ is a local subsolution of $P(g, \Omega)$, we infer $U^+ \le S^+(u, B)$ on D, which clearly contradicts the definition of D. As a consequence, u is a local subsolution of $P(g, \Omega)$, so by definition $U^+ \ge u$ on Ω, which yields $U^+ \ge S^-(U^+, V)$ on V. The function U^+ is thus a supersolution of $P(g, \Omega)$, which concludes the proof. □

Theorem 4.9 is now a direct consequence of the above serie of results. Indeed, U^+ is an AML thanks to propositions 4.15 and 4.16.

Remark 4.17. The same proof as in proposition 4.16 yields that the function $U^- : x \mapsto \inf\{u(x) : u \in S^{+loc}(P(g, \Omega))\}$ is both a local supersolution and a local subsolution of $P(g, \Omega)$. It is thus also an AML of $P(g, \Omega)$, and one has that for every AML u of $P(g, \Omega)$: $U^- \le u \le U^+$.

Remark 4.18. Note that the hypothesis (H_1) is only necessary in our proof of the caracterization of AMLs as local sub- and supersolutions (proposition 4.15)

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