A new class of costs for optimal transport problems

Thierry Champion

Laboratoire IMATH, Université de Toulon

joint work with **G. Bouchitté** and **J.J. Alibert** (IMATH)

Classical optimal transport problem

- X, Y convex, compact sets (in some $\mathbb{R}^d$)
- cost function $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$, lower semicontinuous
- $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ Borel probabilities on X,Y

The **classical Monge-Kantorovich problem** associated to c :

$$(MK) \qquad \inf \left\{ \int_{X,Y} c(x,y) d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\}$$

$\Pi(\mu,\nu)$: set of transport plans from μ to ν

$$\gamma \in \Pi(\mu,\nu) \quad \Leftrightarrow \quad \begin{cases} \forall A, & \gamma(A \times Y) = \mu(A) \\ \forall B, & \gamma(X \times B) = \nu(B) \end{cases}$$

$$\Leftrightarrow \quad \forall \phi, \psi, \quad \int_{X \times Y} \phi(x) + \psi(y) d\gamma = \int_X \phi d\mu + \int_Y \psi d\nu$$

Classical optimal transport problem

Discrete : if $\mu = \sum_i \mu_i \, \delta_{x_i}$ and $\nu = \sum_j \nu_j \, \delta_{y_j}$

then $\gamma = \sum_{i,j} \gamma_{i,j} \, \delta_{(x_i, y_j)}$ belongs to $\Pi(\mu, \nu)$

whenever $\mu_i = \sum_j \gamma_{i,j}$ and $\nu_j = \sum_i \gamma_{i,j}$.

Note : $\gamma_{i,j}$ = amount of mass moved from x_i to y_j.

Classical optimal transport problem

Discrete : if $\mu = \sum_i \mu_i \, \delta_{x_i}$ and $\nu = \sum_j \nu_j \, \delta_{y_j}$

then $\gamma = \sum_{i,j} \gamma_{i,j} \, \delta_{(x_i, y_j)}$ belongs to $\Pi(\mu, \nu)$

whenever $\mu_i = \sum_j \gamma_{i,j}$ and $\nu_j = \sum_i \gamma_{i,j}$.

Note : $\gamma_{i,j}$ = amount of mass moved from x_i to y_j.

Product : $\gamma = \mu \times \nu$ belongs to $\Pi(\mu, \nu)$

Classical optimal transport problem

Transport maps: if $T_{\#}\mu = \nu$ then $(id \times T)_{\#}\mu \in \Pi(\mu, \nu)$.

$$(\forall A, \quad T_{\#}\mu(A) := \mu(T^{-1}(A)))$$

Classical optimal transport problem

Transport maps : if $T_{\#}\mu = \nu$ then $(id \times T)_{\#}\mu \in \Pi(\mu, \nu)$.

$$(\forall A, \quad T_{\#}\mu(A) := \mu(T^{-1}(A)))$$

Discrete : if $\mu = \sum_i \mu_i \, \delta_{x_i}$ and $\nu = \sum_j \nu_j \, \delta_{y_j}$

then $T_{\#}\mu = \nu \quad \Leftrightarrow \quad \forall j, \quad \nu_j = \sum_{i:\, x_i \in T^{-1}(y_j)} \mu_i$

and $(id \times T)_{\#}\mu = \sum_i \mu_i \, \delta_{(x_i, T(x_i))}$

(MK) is the relaxed version of the **Monge problem**

$$(M) \qquad \inf \left\{ \int_X c(x, T(x)) d\mu(x) \; : \; T_{\#}\mu = \nu \right\}$$

References (MK)-(M): Villani (2003,2009), Santambrogio (2015)

Desintegration of γ

Take $\gamma \in \Pi(\mu, \nu)$

Write $\gamma = \gamma^x \otimes \mu$, **desintegration of γ with respect to μ** :

$$\gamma^x \in \mathcal{P}(Y) \quad \mu - a.e.\ x$$

$$\forall f \in \mathcal{C}_b(X \times Y), \qquad \langle \gamma, f \rangle = \int_X \left(\int_Y f(x,y) d\gamma^x(y) \right) d\mu(x)$$

Desintegration of γ

Take $\gamma \in \Pi(\mu, \nu)$

Write $\gamma = \gamma^x \otimes \mu$, **desintegration of γ with respect to μ** :

$$\gamma^x \in \mathcal{P}(Y) \quad \mu - a.e.\ x$$

$$\forall f \in \mathcal{C}_b(X \times Y), \qquad \langle \gamma, f \rangle = \int_X \left(\int_Y f(x,y) d\gamma^x(y) \right) d\mu(x)$$

Discrete :

if $\mu = \sum_i \mu_i \, \delta_{x_i}$, $\nu = \sum_j \nu_j \, \delta_{y_j}$

and $\gamma = \sum_{i,j} \gamma_{i,j} \, \delta_{(x_i, y_j)}$ then $\gamma^{x_i} = \sum_j \frac{\gamma_{i,j}}{\mu_i} \delta_{y_j}$

Desintegration of γ

Take $\gamma \in \Pi(\mu, \nu)$

Write $\gamma = \gamma^x \otimes \mu$, **desintegration of γ with respect to μ** :

$$\gamma^x \in \mathcal{P}(Y) \quad \mu - a.e.\ x$$

$$\forall f \in \mathcal{C}_b(X \times Y), \qquad \langle \gamma, f \rangle = \int_X \left(\int_Y f(x,y) d\gamma^x(y) \right) d\mu(x)$$

Discrete : if $\mu = \sum_i \mu_i \, \delta_{x_i}$, $\nu = \sum_j \nu_j \, \delta_{y_j}$

and $\gamma = \sum_{i,j} \gamma_{i,j} \, \delta_{(x_i, y_j)}$ then $\gamma^{x_i} = \sum_j \frac{\gamma_{i,j}}{\mu_i} \delta_{y_j}$

Transport map : if $\gamma = (id \times T)_{\#}\mu$ then $\gamma^x = \delta_{T(x)}$ $a.e.\ x$

Desintegration of γ

Take $\gamma \in \Pi(\mu, \nu)$

Write $\gamma = \gamma^x \otimes \mu$, **desintegration of γ with respect to μ** :

$$\gamma^x \in \mathcal{P}(Y) \quad \mu - a.e.\ x$$

$$\forall f \in \mathcal{C}_b(X \times Y), \qquad \langle \gamma, f \rangle = \int_X \left(\int_Y f(x,y) d\gamma^x(y) \right) d\mu(x)$$

Discrete : if $\mu = \sum_i \mu_i \, \delta_{x_i}$, $\nu = \sum_j \nu_j \, \delta_{y_j}$

and $\gamma = \sum_{i,j} \gamma_{i,j} \, \delta_{(x_i, y_j)}$ then $\gamma^{x_i} = \sum_j \frac{\gamma_{i,j}}{\mu_i} \delta_{y_j}$

Transport map : if $\gamma = (id \times T)_{\#}\mu$ then $\gamma^x = \delta_{T(x)}$ $a.e.\ x$

Product : if $\gamma = \mu \times \nu$ then $\gamma^x = \nu$ $a.e.\ x$

Desintegration of γ

The **classical Monge-Kantorovich problem** now reads :

$$(MK) \qquad \inf \left\{ \int_X \int_Y c(x,y) d\gamma^x(y) d\mu(x) \ : \ \int_X \gamma^x d\mu(x) = \nu \right\}$$

About $\displaystyle\int_X \gamma^x d\mu(x) = \nu$:

Discrete : if $\displaystyle \mu = \sum_i \mu_i \, \delta_{x_i} \ , \quad \nu = \sum_j \nu_j \, \delta_{y_j} \ , \quad \gamma \in \Pi(\mu, \nu),$

then $\displaystyle \gamma^{x_i} = \sum_j \frac{\gamma_{i,j}}{\mu_i} \delta_{y_j}$ with $\displaystyle \nu_j = \sum_i \gamma_{i,j}$

and $\displaystyle \int_X \gamma^x d\mu(x) = \sum_i \left(\mu_i \sum_j \frac{\gamma_{i,j}}{\mu_i} \delta_{y_j} \right) = \nu$

Desintegration of γ

The **classical Monge-Kantorovich problem** now reads :

$$(MK) \qquad \inf \left\{ \int_X \int_Y c(x,y) d\gamma^x(y) d\mu(x) \ : \ \int_X \gamma^x d\mu(x) = \nu \right\}$$

About $\displaystyle\int_X \gamma^x d\mu(x) = \nu$:

Continuous : if $\nu = \nu(y)dy$ and $\gamma^x = \gamma^x(y)dy$ for a.e. x

then $$\int_X \gamma^x(y) d\mu(x) = \nu(y) \quad \text{for a.e. } x.$$

Desintegration of γ

The **classical Monge-Kantorovich problem** now reads :

$$(MK) \qquad \inf\left\{\int_X\int_Y c(x,y)d\gamma^x(y)d\mu(x) \;:\; \int_X \gamma^x d\mu(x) = \nu\right\}$$

can be rewritten

$$(MK) \qquad \inf\left\{\int_X G(x,\gamma^x)d\mu(x) \;:\; \int_X \gamma^x d\mu(x) = \nu\right\}$$

with $\quad G : (x,p) \in X \times \mathcal{P}(Y) \mapsto G(x,p) = \int_Y c(x,y)dp(y)$

Note : G is linear in p.

New class of costs

In this talk, we are interested in the generalization of (MK) :

$$\mathcal{F}(\mu,\nu) \;=\; \inf\left\{\int_X G(x,\gamma^x)d\mu(x) \,:\, \int_X \gamma^x d\mu(x) = \nu\right\}$$

with

$$G : (x,p) \in X \times \mathcal{P}(Y) \mapsto G(x,p) = \int_Y c(x,y)dp(y) + H(x,p)$$

where $H : X \times \mathcal{P}(Y) \to [0,+\infty]$ is a entropy / perturbation cost.

New class of costs

- **Cardinal cost** $H(x, p) = \#(support(p)) - 1$.
  Note :

  $$H(x, \gamma^x) = 0 \text{ a.e.} \Leftrightarrow \gamma^x = \delta_{T(x)} \text{ a.e.} \Leftrightarrow \gamma = (id \times T)\#\mu$$

  so that $\mathcal{F}(\mu, \nu) = \inf(M) = \min(MK)$ when μ has no atoms.

  Then $\mathcal{F}(\mu, \nu)$ may have no solution despite H is l.s.c. on $\mathcal{P}(Y)$.

New class of costs

- **"Variance" cost** $H(x,p) = var(p) = \int_Y |y|^2 dp(y) - |[p]|^2$

where $[p] = \int_Y y\, dp(y)$ denotes the barycenter of p. Note

$$\int_X H(x,\gamma^x) d\mu(x) = \int_Y |y|^2 d\nu(y) - \int_X |[\gamma^x]|^2 d\mu(x)$$

H is not convex in p, $\mathcal{F}(\mu,\nu)$ may have no solution.

New class of costs

- **"Variance" cost** $H(x,p) = var(p) = \int_Y |y|^2 dp(y) - |[p]|^2$

  where $[p] = \int_Y y\, dp(y)$ denotes the barycenter of p. Note

  $$\int_X H(x,\gamma^x) d\mu(x) = \int_Y |y|^2 d\nu(y) - \int_X |[\gamma^x]|^2 d\mu(x)$$

  H is not convex in p, $\mathcal{F}(\mu,\nu)$ may have no solution.

- **Variance cost** $H(x,p) = -var(p)$ or $H(x,p) = |[p]|^2$.

  Then H is l.s.c. and convex on $\mathcal{P}(Y)$.

  H favours the spreading of p (max. of variance).

New class of costs

- **Barycenter constraint**

$$H(x,p) = \chi_{[p]=x} = \begin{cases} 0 & \text{if } [p] = x \\ +\infty & \text{otherwise} \end{cases}$$

For the cost $c(x,y) = -|y-x|$, $\mathcal{F}(\mu,\nu)$ is related to model-independent pricing in mathematical finance [Hobson Neuberger 2012] and [Beiglböck Henry-Labordère Penkner 2013].

Existence of a particular solution γ : [Beiglböck Juillet –]

Note : $\mathcal{F}(\mu,\nu) < +\infty \quad \Leftrightarrow \quad \mu \preceq \nu$ for convex order

Existence result

Main hypotheses

(H_1) $c : X \times Y \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous,

(H_2) $H : X \times \mathcal{P}(Y) \to \mathbb{R} \cup \{+\infty\}$ satisfies

- H is lower semicontinuous on $X \times \mathcal{P}(Y)$.
- for every $x \in X$, $p \mapsto H(x, p)$ is convex.

Theorem

Assume (H_1) and (H_2), and recall

$$\mathcal{F}(\mu, \nu) \;=\; \inf \left\{ \int_X G(x, \gamma^x) d\mu(x) \;:\; \int_X \gamma^x d\mu(x) = \nu \right\}$$

then $\mathcal{F}$ is lower semicontinuous on $\mathcal{M}_b^+(X) \times \mathcal{M}_b^+(Y)$.

Moreover, if $\mathcal{F}(\mu, \nu) < +\infty$ then there is at least one minimizer.

$\mathcal{F}(\mu, \nu)$ extended by $\mathcal{F}(\mu, \nu) = +\infty$ whenever $\mu(X) \neq \nu(Y)$

Lower semicontinuity property

Set $\quad E(\gamma) = \int_X G(x,\gamma^x)d\mu \quad$ whenever $\gamma \in \Pi(\mu,\nu)$

Lemma – *Lower semicontinuity of E*

Assume (H_1) and (H_2), $(\gamma_n)_n = (\gamma_n^x \otimes \mu_n)_n$ weakly converges in $\mathcal{M}_b(X \times Y)$ to $\gamma = \gamma^x \otimes \mu$,

$$\text{then} \qquad \liminf_{n\to+\infty} \int_X G(x,\gamma_n^x)d\mu_n \geq \int_X G(x,\gamma^x)d\mu.$$

Note : convexity of $p \mapsto H(x,p)$ is necessary

counterexamples follow for cardinal cost

$$H(x,p) = \#(support(p)) - 1$$

when $\inf(M) = \min(MK)$ and (M) not attained

Lower semicontinuity property

Let $G^*(x,\cdot)$ denote the Fenchel conjugate of the convex $G(x,\cdot)$:

$$\forall \psi \in \mathcal{C}(Y) \qquad G^*(x,\psi) = \sup\left\{ \int_Y \psi dp - G(x,p) : p \in \mathcal{P}(Y) \right\}.$$

Then one has :

- *Upper semicontinuity* : if $\psi \in \mathcal{C}(Y)$ then

$$x \mapsto G^*(x,\psi) \quad \text{is upper semicontinuous}$$

- *bounds* : denote $\quad m_G = \inf G \quad$ then

$$\inf_Y \psi - m_G \leq G^*(x,\psi) \leq \sup_Y \psi - m_G$$

- *Lipschitz property* For every $x \in X$, $G^*(x,\cdot)$ satisfies

$$|G^*(x,\psi_1) - G^*(x,\psi_2)| \;\leq\; \sup_Y |\psi_1 - \psi_2| \,.$$

Lower semicontinuity property

Let $(\psi_k)_k$ a dense sequence in $\mathcal{C}(Y)$.

Since $G(x,\cdot)$ convex l.s.c. :

$$\forall p \in \mathcal{P}(Y), \quad G(x,p) = \sup_k \int \psi_k dp - G^*(x,\psi_k) = \sup_k G_k(x,p)$$

Then for $(\Omega_k)_{1\leq k\leq m}$ disjoint open sests

$$\begin{aligned}
\int_X G(x,\gamma_n^x) d\mu_n(x) &\geq \sum_{k=0}^m \int_{\Omega_k} G_k(x,\gamma_n^x) d\mu_n(x) \\
&= \sum_{k=0}^m \Big[\int_{\Omega_k \times Y} \psi_k(y) d\gamma_n(x,y) \\
&\qquad + \int_{\Omega_k} -G^*(x,\psi_k) d\mu_n(x) \Big]
\end{aligned}$$

Lower semicontinuity property

Then one gets

$$\begin{aligned}\liminf_{n\to+\infty}\int_X G(x,\gamma_n^x)d\mu_n(x) &\geq \sum_{k=0}^m \Big[\int_{\Omega_k\times Y}\psi_k(y)d\gamma(x,y) \\ &\qquad + \int_{\Omega_k} -G^*(x,\psi_k)d\mu(x)\Big] \\ &= \sum_{k=0}^m \int_{\Omega_k} G_k(x,\gamma^x)d\mu_n(x)\end{aligned}$$

Taking the sup on m and the open partitions yields :

$$\liminf_{n\to+\infty}\int_X G(x,\gamma_n^x)d\mu_n(x) \geq \int_X G(x,\gamma^x)d\mu(x).$$

Dual problem and optimality conditions

Recall

$$\mathcal{F}(\mu,\nu) \;=\; \inf\left\{\int_X G(x,\gamma^x)d\mu(x)\;:\;\int_X \gamma^x d\mu(x)=\nu\right\}$$

extended by 1-homogeneity on $\mathcal{M}_b^+(X)\times\mathcal{M}_b^+(Y)$.

From convexity and lower-semicontinuity it comes

Assume (H_1) and (H_2), then

$$\mathcal{F}(\mu,\nu) \;=\; \sup\left\{\int_Y \psi(y)d\nu - \int_X G^*(x,\psi)d\mu(x)\;:\;\psi\in\mathcal{C}^0(Y)\right\}$$

and equality holds in $[0,+\infty]$.

Moreover the dual pair (γ,ψ) is optimal whenever

$$\psi\in\partial G(x,\gamma^x)\quad \mu-a.e.$$

Dual problem and optimality conditions

- if $H = 0$, then $\quad G(x,p) = \int_Y c(x,y)dp(y),$

$$G^*(x,\psi) = \sup_{p \in \mathcal{P}(Y)} \int_Y \psi(y) - c(x,y)\,dp = \sup_{y \in Y} \psi(y) - c(x,y) = -\psi^c(x)$$

so that one recovers the classical Kantorovich dual problem

$$\mathcal{F}(\mu,\nu) \;=\; \sup\left\{ \int_Y \psi(y)d\nu + \int_X \psi^c(x)d\mu(x) \;:\; \psi \in \mathcal{C}^0(Y) \right\}$$

Dual problem and optimality conditions

- if $c = 0$ and $H(x,p) = \chi_{[p]=x}$ then

$$G^*(x,\psi) = \sup_{p \in \mathcal{P}(Y)} \left\{ \int_Y \psi(y) dp : [p] = x \right\}$$

$$= - \inf_{p \in \mathcal{P}(Y)} \left\{ \int_Y -\psi(y) dp : [p] = x \right\} = -(-\psi)^{**}(x)$$

so that (here $X = Y$) :

$$\mathcal{F}(\mu,\nu) = \sup \left\{ - \int_X -\psi d\nu + \int_X (-\psi)^{**} d\mu(x) \; : \; \psi \in \mathcal{C}^0(X) \right\}$$

and then we recover [Strassen 1965]

$$\mathcal{F}(\mu,\nu) < +\infty \quad \Leftrightarrow \quad \int \psi d\mu \leq \int \psi d\nu \qquad \forall \psi \text{ convex}$$

Dual problem and optimality conditions

- if $\quad H(x,p) = h(x,[p]) \quad$ then

$$G^*(x,\psi) \; = \; -\inf_{z\in\mathbb{R}^d} \{(c(x,\cdot)-\psi)^{**}(-z) + h(x,z)\}$$

and the optimality condition reads : for $\mu - a.e.x$

$$0 \in \partial h(x,[\gamma^x]) + \partial(c(x,\cdot)-\psi)^{**}([\gamma^x])$$

and $\quad \displaystyle\int_\Omega (c(x,y)-\psi(y))\gamma^x(dy) = (c(x,\cdot)-\psi)^{**}([\gamma^x])$

Dual problem and optimality conditions

- if $H(x,p) = h(x,[p])$ then

$$G^*(x,\psi) \;=\; -\inf_{z\in\mathbb{R}^d} \{(c(x,\cdot)-\psi)^{**}(-z) + h(x,z)\}$$

and the optimality condition reads : for $\mu - a.e.x$

$$0 \in \partial h(x,[\gamma^x]) + \partial(c(x,\cdot)-\psi)^{**}([\gamma^x])$$

and $$\int_\Omega (c(x,y)-\psi(y))\gamma^x(dy) = (c(x,\cdot)-\psi)^{**}([\gamma^x])$$

model case : $h(x,[p]) = \chi_{[p]=x}$

then $G^*(x,\psi) = -(c(x,\cdot)-\psi(\cdot))^{**}(x)$

Dual problem and optimality conditions

- if $H(x,p) = h(x,[p])$ then

$$G^*(x,\psi) \;=\; -\inf_{z\in\mathbb{R}^d} \{(c(x,\cdot)-\psi)^{**}(-z) + h(x,z)\}$$

and the optimality condition reads : for $\mu - a.e.x$

$$0 \in \partial h(x,[\gamma^x]) + \partial(c(x,\cdot)-\psi)^{**}([\gamma^x])$$

and $$\int_\Omega (c(x,y)-\psi(y))\gamma^x(dy) = (c(x,\cdot)-\psi)^{**}([\gamma^x])$$

model case : $c(x,y) = \lambda|y-x|^2, \quad h(x,[p]) = \lambda|[p]|^2$

then $G^*(x,\psi) = -(|\cdot| - \psi)^{**} \triangledown |\cdot|(\lambda x) - \lambda(1-\lambda)|x|^2$

Existence of a solution ψ for dual problem

Difficult task : in [Beiglböck Henry-Labordère Penkner 2013]

counterexample for $c(x, y) = -|y - x|$ and $H(x, p) = \chi_{[p]=x}$

for some discrete μ on $[0, 2]$ and $\nu = \frac{1}{2} dx_{\lfloor [0,2]}$

Existence of a solution ψ for dual problem

Difficult task : in [Beiglböck Henry-Labordère Penkner 2013]

counterexample for $c(x,y) = -|y-x|$ and $H(x,p) = \chi_{[p]=x}$

for some discrete μ on $[0,2]$ and $\nu = \frac{1}{2}dx_{\lfloor[0,2]}$

back to classical dual case :

$$\begin{aligned} & \sup\left\{\int_Y \psi(y)d\nu + \int_X \psi^c(x)d\mu(x) \,:\, \psi \in \mathcal{C}^0(Y)\right\} \\ = \; & \sup\left\{\int_Y (\psi^c)^c(y)d\nu + \int_X \psi^c(x)d\mu(x) \,:\, \psi \in \mathcal{C}^0(Y)\right\} \end{aligned}$$

with $\psi^c(x) = \sup_{y\in Y}\{\psi(y) - c(x,y)\}$ and $((\psi^c)^c)^c = \psi^c$.

Existence of a solution ψ for dual problem

classical dual case, c subadditive $(c(x,z) \leq c(x,y) + c(y,z))$:

$$\sup\left\{\int_Y (\psi^c)^c(y)d\nu + \int_X \psi^c(x)d\mu(x) \,:\, \psi \in \mathcal{C}^0(Y)\right\}$$

$$\sup\left\{\int_Y \psi^c(y)d\nu - \int_X \psi^c(x)d\mu(x) \,:\, \psi \in \mathcal{C}^0(Y)\right\}$$

i.e. $(\psi^c)^c = \psi^c$ $\quad\rightarrow$ look for a solution of the form ψ^c.

framework : $X = Y$, c subadditive $(c(x,z) \leq c(x,y) + c(y,z))$

goal : find conditions for which $G^*(\cdot, G^*(\cdot,\psi)) = G^*(\cdot,\psi)$.

First : if $c(x,x) = 0$ and $H(x,\delta_x) = 0$ then $G(x,\delta_x) = 0$

so that $G^*(x,\psi) \geq \psi(x)$ for all x, ψ

Existence of a solution ψ for dual problem

Proposition

if c subadditive, and

$H(x,p) = h([p])$ with h l.s.c. convex, $h(0) = 0$, $h \geq 0$,

or $H(x,p) = h([p] - x)$ h as above + subadditive,

then $G^*(\cdot, G^*(\cdot, \psi)) = G^*(\cdot, \psi)$ for any $\psi \in \mathcal{C}(X)$.

Applies in particular to $H(p) = |[p]|^2$ and $H(x,p) = \chi_{[p]=x}$.

Example : barycenter constraint

Take $c(x,y) = |y-x|$ and $H(x,p) = \chi_{[p]=x}$

set $\mu = \frac{1}{2} dx_{\lfloor[-1,1]}$ and $\nu = \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1$.

Then $\gamma^x = \frac{|x|-x}{2}\delta_{-1} + (1-|x|)\delta_0 + \frac{|x|+x}{2}\delta_1$

and $\mathcal{F}(\mu,\nu) = \frac{1}{3}$ while $\inf(M) = \frac{1}{4}$

Set $\psi(0) = 0$, then

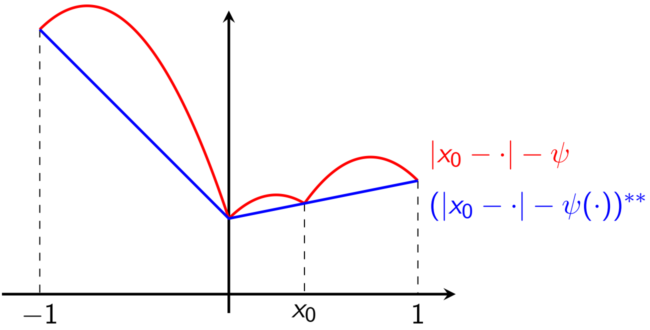
$$\int_\Omega (c(x,y) - \psi(y))\gamma^x(dy) = (c(x,\cdot) - \psi)^{**}([\gamma^x]) = -G^*(x,\psi)$$

implies
$\psi(y) = -(|y-\cdot| - \psi)^{**}(x) = 2x(1+x) - \psi(-1)x$ if $x \leq 0$,
$\psi(y) = -(|y-\cdot| - \psi)^{**}(x) = 2x(x-1) + \psi(1)x$ if $x \geq 0$.

Example : barycenter constraint

Set $\psi(1) = \psi(-1) = 0$ then a solution of the dual problem is

$$\psi(x) = \begin{cases} 2x(x+1) & \text{if } x \leq 0, \\ 2x(x-1) & \text{if } x \geq 0. \end{cases}$$



$$\int_\Omega (c(x_0, y) - \psi(y))\gamma^{x_0}(dy) = (c(x_0, \cdot) - \psi)^{**}(x_0) = -G^*(x_0, \psi)$$

Example : variance cost

Take $c(x,y)=\lambda|y-x|^2$ and $H(x,p)=|[p]|^2$

set $\mu=\frac{1}{2}\delta_0+\frac{1}{2}\delta_1$ and $\nu=dx_{\lfloor[0,1]}$.

Then for $\lambda\geq\frac{1}{2}$, $\gamma^0=dx_{\lfloor[0,\frac{1}{2}]}$ and $\gamma^1=dx_{\lfloor[\frac{1}{2},1]}$