Approachability Theory and Differential Games

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Approachability theory

: Blackwell (1956)

Given an $I \times J$ matrix A with coefficients in R^k, a two-person infinitely repeated game form G is defined as follows.

At each stage $n = 1, 2, ...,$ each player chooses an element in his set of actions: $i_n \in I$ for Player 1 (resp. $j_n \in J$ for Player 2), the corresponding outcome is $g_n = A_{i_n j_n} \in R^k$ and the couple of actions (i_n, j_n) is announced to both players.

$\overline{g}_n = \frac{1}{n} \sum_{m=1}^{n} g_m$ is the average outcome up to stage n.

The aim of Player 1 is that $\overline{g}_n$ approaches a target set $C \subset R^k$.

Approachability: generalization of max-min level in a (one shot) game with real payoff where $C = [v, +\infty)$.

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$H_n = (I \times J)^n$ is the set of possible histories at stage $n+1$ and $H_\infty = (I \times J)^\infty$ be the set of plays.

Σ (resp. $\mathcal{T}$) is the set of strategies of Player 1 (resp. Player 2): mappings from $H = \cup_{n \geq 0} H_n$ to the sets of mixed actions $U = \Delta(I)$ (probabilities on I) (resp. $V = \Delta(J)$).

At stage n, given the history $h_{n-1} \in H_{n-1}$, Player 1 chooses an action in I according to the probability distribution $\sigma(h_{n-1}) \in U$ (and similarly for Player 2).

A couple (σ, τ) of strategies induces a probability on H_∞ and $\mathsf{E}_{\sigma,\tau}$ denotes the corresponding expectation.

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- asympotic analysis: study of the limit of **values** of finitely repeated games or discounted games as the expected length goes to ∞.

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Definition

A nonempty closed set C in $\mathbf{R}^k$ is **weakly approachable** by Player 1 in G if, for every $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for any $n \geq N$ there is a strategy $\sigma = \sigma(n, \varepsilon)$ of Player 1 such that, for any strategy τ of Player 2

$$\mathrm{E}_{\sigma,\tau}(d_C(\bar{g}_n)) \leq \varepsilon.$$

where d_C stands for the distance to C.

If v_n is the value of the n-stage game with payoff $-E(d_C(\bar{g}_n))$, weak-approachability means $v_n \to 0$.

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2.1 The “expected deterministic” repeated game form G^*

Alternative two-person infinitely repeated game associated, as the previous one, to the matrix A.

At each stage $n = 1, 2, ...$, Player 1 (resp. Player 2) chooses $u_n \in U = \Delta(I)$ (resp. $v_n \in V = \Delta(J)$), the outcome is $g^*_n = u_n A v_n$ and (u_n, v_n) is announced.

Accordingly, a strategy σ^* for Player 1 in G^* is a map from H^* $=\cup_{n\geq 0} H^*_n$ to U where $H^*_n = (U \times V)^n$. A strategy τ^* for Player 2 is defined similarly.

A couple of strategies induces a play $\{(u_n, v_n)\}$ and a sequence of outcomes $\{g^*_n\}$, and $\overline{g}^*_n = \frac{1}{n}\sum_{m=1}^{n} g^*_m$ denotes the average outcome up to stage n.

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2.2 Differential games

Consider zero-sum differential games of the kind

$$\dot{x} = f(x, u, v)$$

where x is the state and u, v the moves.
Assume

$$\begin{cases} (i) & U, \quad V \quad \text{are compact subsets of } \ \mathbf{R}^k, \\ (ii) & f : \mathbf{R}^k \times U \times V \mapsto \mathbf{R}^k \quad \text{is continuous}, \\ (iii) & f(., u, v) \ \text{ is a l- Lipschitz map for all } (u, v) \in U \times V, \\ (iv) & U \ \text{ is convex, and } f \text{ is affine in } \ u. \end{cases} \tag{1}$$

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Sets of controls :
$\mathbf{U} = \{\mathbf{u} : [0, +\infty) \mapsto U; \mathbf{u}$ is measurable$\}$ and similarly $\mathbf{V}$.
Induced dynamics with $x_0 \in R^k$ and $(\mathbf{u}, \mathbf{v}) \in \mathbf{U} \times \mathbf{V}$:

$$\begin{cases} \dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), \mathbf{v}(t)) & \text{for almost every } \; t \geq 0 \\ \mathbf{x}(0) = x_0. \end{cases} \tag{2}$$

In addition Isaacs condition, namely : for any $\zeta \in R^k$

$$\sup_{v \in V} \inf_{u \in U} \langle \zeta, f(x, u, v) \rangle = \inf_{u \in U} \sup_{v \in V} \langle \zeta, f(x, u, v) \rangle. \tag{3}$$

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(Vieille, 1992)

The aim is to obtain a good average outcome at stage n.
First consider the game G^*. Then use as state variable the cumulative payoff and consider the differential game Λ of fixed duration played on $[0, 1]$ starting from $\mathbf{x}(0) = 0$ with dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t)A\mathbf{v}(t)$$

and payoff $-d_C(\mathbf{x}(1))$.
The state variable is $x(t) = \int_0^t g_s ds$ with g_s being the payoff at time s. G_n^* appears then as a discrete time approximation of Λ.
Let $\Phi(t, x)$ be the value of the game played on $[t, 1]$ starting from x (i.e. with total outcome $x + \int_t^1 g_s ds$).

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Theorem

1) $\Phi(x,t)$ *is the unique viscosity solution of*

$$\frac{d}{dt}\Phi(x,t) + \mathtt{val}_{U\times V}\langle \nabla\Phi(x,t), uAv\rangle = 0$$

with $\Phi(x,1) = -d_C(x)$.
2)

$$\lim v_n^* = \Phi(0,0)$$

The last step is to relate the values in G_n^* and in G_n.

Theorem

$$\lim v_n^* = \lim v_n$$

Consider an optimal strategy in G_n^*.
Each stage m in this game will correspond to a block of L stages in G_{Ln} where player 1 will play i.i.d. the prescribed strategy in G^* and will define inductively y_m^* as the empirical distribution of moves of Player 2 during this block.

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Each stage m in this game will correspond to a block of L stages in G_{Ln} where player 1 will play i.i.d. the prescribed strategy in G^* and will define inductively y_m^* as the empirical distribution of moves of Player 2 during this block.

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Every set is weakly approachable or excludable.

The main notion was introduced by Blackwell:

Definition

A closed set C in $\mathbf{R}^k$ is a **B**-set for Player 1 (given A), if for any $z \notin C$, there exists $y \in \pi_C(z)$ and a mixed action $u = \hat{u}(z)$ in $U = \Delta(I)$ such that the hyperplane through y orthogonal to the segment $[yz]$ separates z from uAV:

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The basic Blackwell's result is :

Theorem

Let C be a **B***-set for Player 1. Then it is approachable in G and* $*$*approachable in* G^* *by that player. An approchability strategy is given by* $\sigma(h_n) = \hat{u}(\bar{g}_n)$ *(resp.* $\sigma^*(h_n^*) = \hat{u}(\bar{g}_n^*)$*).*

The proof for approachability is Proposition 8 in [1]. The other one is a simple adaptation where the outcome $\bar{g}_n$ is replaced by $\bar{g}_n^*$.

Remark. The previous Proposition implies that a **B**-set remains approachable (resp. $*$approachable) in the game where the only information of Player 1 after stage n is the current outcome g_n (resp. g_n^*) rather than the complete previous history h_n (resp. h_n^*). (natural state variable)

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An important consequence of this property is

Theorem

A convex set C is either approachable or excludable.

A further result due to Spinat [11] characterizes minimal approachable sets:

Theorem

A set C is approachable iff it contains a **B***-set.*

To study $*$approachability, we introduce an allternative differential game Γ where both the dynamics and the payoff differs from the previous differential game Λ.

The aim is to control the average payoff hence the discrete dynamics on the state variable is of the form

$$\bar{g}_{n+1} - \bar{g}_n = \frac{1}{n+1}(g_{n+1} - \bar{g}_n)$$

Continuous counterpart is $\gamma(\mathbf{u}, \mathbf{v})(t) = \frac{1}{t}\int_0^t \mathbf{u}(s)A\mathbf{v}(s)ds$.
Change of variable $\mathbf{x}(s) = \gamma(e^s)$ leads to

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t)A\mathbf{v}(t) - \mathbf{x}(t). \tag{4}$$

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In addition the aim of Player 1 is to stay in a certain set C.

We introduce the following definitions:

Definition

A map $\alpha : \mathbf{V} \to \mathbf{U}$ is a **nonanticipative strategy** if, for any $t \geq 0$ and for any $\mathbf{v}_1$ and $\mathbf{v}_2$ of $\mathcal{V}$, which coincide almost everywhere on $[0, t]$ of $[0, +\infty)$, the images $\alpha(\mathbf{v}_1)$ and $\alpha(\mathbf{v}_2)$ coincide also almost everywhere on $[0, t]$.

$\mathcal{M}(\mathbf{V}, \mathbf{U})$ is the set of nonanticipative strategies from $\mathbf{V}$ to $\mathbf{U}$.

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Definition

A non-empty closed set C in $\mathbf{R}^k$ is a **discriminating domain** for Player 1, given f if:

$$\forall x \in C, \quad \forall p \in NP_C(x), \qquad \sup_{v \in V} \inf_{u \in U} \langle f(x,u,v), p \rangle \leq 0, \qquad (5)$$

where $NP_C(x)$ is the set of proximal normals to C at x

$$NP_C(x) = \{p \in \mathbf{R}^K ; d_C(x+p) = \|p\|\}$$

The interpretation is that, at any boundary point $x \in C$, Player 1 can react to any control of Player 2 in order to keep the trajectory in the half space facing a proximal normal p.

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The following theorem, due to Cardaliaguet [2], states that Player 1 can ensure remaining in a discriminating domain as soon as he knows, at each time t, Player 2's control up to time t.

Theorem

Assume that f satisfies conditions (1), and that C is a closed subset of $\mathbb{R}^k$. Then C is a discriminating domain if and only if for every x_0 belonging to C, there exists a nonanticipative strategy $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$, such that for any $\mathbf{v} \in \mathbf{V}$, the solution $\mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t)$ remains in C for every $t \geq 0$.

We shall say that such a strategy α **preserves** the set C.

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Theorem

Let $f(x,u,v) = uAv - x$. A closed set $C \subset \mathbf{R}^k$ is a discriminating domain for Player 1, if and only if C is a **B***-set for Player 1.*

First condition: start from z, $x = \pi_C(z)$, there exists u such that for all v

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It is easy to deduce that starting from any point, not necessarily in C one has:

Theorem

If a closed set $C \subset R^k$ is a **B***-set for Player 1, there exists a nonanticipative strategy of player 1 in* Γ, $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$, *such that for every* $\mathbf{v} \in \mathbf{V}$

$$\forall t \geq 1 \quad d_C(\mathbf{x}[\alpha(\mathbf{v}), \mathbf{v}](t)) \leq Me^{-t}. \tag{6}$$

Theorem

A closed set C is $*$approachable for Player 1 in G^* if and only if it contains a **B***-set for Player 1 (given A).*

The direct part follows from Blackwell's proof.
To obtain the converse implication, the proof follows several steps:
First, we construct a map Ψ from strategies of Player 1 in G^* to nonanticipative strategies in Γ.
In particular given $\varepsilon > 0$ and a strategy σ_ε that ε-approaches C in G^*, we define its image $\alpha_\varepsilon = \Psi(\sigma_\varepsilon)$.
The next step consists in proving that the trajectories in the differential game Γ compatible with α_ε approach asymptotically $C + \varepsilon\bar{B}$.

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Then, we show that the ω-limit set of any trajectory compatible with some $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$ is a nonempty compact discriminating domain for f.

Explicitely, let

$$D(\alpha) = \bigcap_{\theta \geq 0} cl\{\mathbf{x}[x_0, \alpha(\mathbf{w}), \mathbf{w}](t); \quad t \geq \theta, \ \ \mathbf{w} \in \mathbf{V}\}.$$

(where cl is the closure operator).

Lemma

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Recall that Spinat [11] proved that a closed set C is approachable in G if and only if it contains a **B**-set, hence we deduce the following corollary.

Corollary

Approachability and $*$approachability coincide.

First introduce a new notion of strategies crucial for time discretization.

Definition

A map $\delta : \mathbf{V} \mapsto \mathbf{U}$ is a **nonanticipative strategy with delay (NAD)** if there exits a sequence of times $0 < t_1 < t_2 < .. < t_n < ..$ going to ∞ with the following property : For every control $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{U}$ such that

$$\mathbf{v}_1(s) = \mathbf{v}_2(s) \text{ for almost every } s \in [0, t_i]$$
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Denote by $\mathcal{M}_d(\mathbf{V}, \mathbf{U})$ the set of such nonanticipative strategies with delay from $\mathbf{V}$ to $\mathbf{U}$.

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We establish a link between preserving NA strategies in the differential game Γ and approachability strategies in the repeated game G.

The idea of the construction is the following:

a) Given a NA strategy α, show that it can be approximated in term of range by a NAD strategy $\bar{\alpha}$.

b) When applied to α preserving C (hence approaching C), obtain a NAD strategy $\bar{\alpha}$ approaching C.

c) This NAD strategy $\bar{\alpha}$ produces an $*$approachability strategy in the repeated game G^*.

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Range associated to a nonanticipative strategy $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$:

$$R(\alpha, t) = cl\{y \in R^k \; \exists \mathbf{v} \in \mathbf{V}, \; y = \mathbf{x}[x_0, \alpha(\mathbf{v}), \mathbf{v}](t)\}.$$

The next result is due to Cardaliaguet ([4]) and is inspired by the "extremal aiming" method of Krasowkii and Subbotin [9], and is very much in the spirit of proximal normals and approachability.

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Proposition

Consider the differential game (2). For any $\varepsilon > 0$, $T > 0$ and any nonanticipative strategy $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$, there exists some nonanticipative strategy with delay $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ such that, for all $t \in [0, T]$ and all $\mathbf{v} \in \mathbf{V}$:

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon.$$

Assume that x_k does not belong to $R(\alpha, t_k)$. Then there exists some control $\mathbf{v}_k \in \mathcal{V}$ such that $y_k := \mathbf{x}[t_0, x_0, \alpha(\mathbf{v}_k), \mathbf{v}_k](t_k)$ is an approximate closest point to x_k in $R(\alpha, t_k)$.

Note $p_k := x_k - y_k$ and take $u_k \in U$ such that

$$\sup_{v \in V} < f(x_k, u_k, v), p_k >= \inf_{u \in U} \sup_{v \in V} < f(x_k, u, v), p_k >= A_k \, . \quad (7)$$

In words, u_k is optimal in the local game at x_k in direction p_k.

Proposition

Consider the differential game (2). For any $\varepsilon > 0$, $T > 0$ and any nonanticipative strategy $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$, there exists some nonanticipative strategy with delay $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ such that, for all $t \in [0, T]$ and all $\mathbf{v} \in \mathbf{V}$:

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon.$$

Assume that x_k does not belong to $R(\alpha, t_k)$. Then there exists some control $\mathbf{v}_k \in \mathcal{V}$ such that $y_k := \mathbf{x}[t_0, x_0, \alpha(\mathbf{v}_k), \mathbf{v}_k](t_k)$ is an approximate closest point to x_k in $R(\alpha, t_k)$.
Note $p_k := x_k - y_k$ and take $u_k \in U$ such that

$$\sup_{v \in V} < f(x_k, u_k, v), p_k >= \inf_{u \in U} \sup_{v \in V} < f(x_k, u, v), p_k >= A_k \ . \quad (7)$$

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The next result relies explicitely on the specific form (4) of the dynamics f in Γ and extends the approximation from a compact interval to to $\boldsymbol{R}^{+}$.

Proposition

Fix $x_0 \in \boldsymbol{R}^k$. For any $\varepsilon > 0$ and any nonanticipative strategy $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$ in the game Γ, there is some nonanticipative strategy with delay $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ such that, for all $t \geq 0$ and all $\mathbf{v} \in \mathbf{V}$:

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The next result relies explicitely on the specific form (4) of the dynamics f in Γ and extends the approximation from a compact interval to to $\boldsymbol{R}^+$.

Proposition

Fix $x_0 \in \boldsymbol{R}^k$. *For any* $\varepsilon > 0$ *and any nonanticipative strategy* $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$ *in the game* Γ, *there is some nonanticipative strategy with delay* $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ *such that, for all* $t \geq 0$ *and all* $\mathbf{v} \in \mathbf{V}$:

$$d_{R(\alpha,t)}(\mathbf{x}[x_0, \overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon.$$

In particular, step b)

Proposition

Let C be a **B***-set. For any $\varepsilon > 0$ there is some nonanticipative strategy with delay $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ in the game Γ and some T such that for any* **v** *in* **V**

$$d_C(\gamma[\overline{\alpha}(\mathbf{v}), \mathbf{v}](t)) \leq \varepsilon, \qquad \forall t \geq T.$$

step c)

Proposition

For any $\varepsilon > 0$ *and any nonanticipative strategy* $\alpha \in \mathcal{M}(\mathbf{V}, \mathbf{U})$ *preserving* C *in the game* Γ*, there is some nonanticipative strategy with delay* $\overline{\alpha} \in \mathcal{M}_d(\mathbf{V}, \mathbf{U})$ *that induces an* ε*-approachability strategy* σ^* *for* C *in* G^**.*

Idea is to use the delay to define a strategy that depens only on the past moves.

The last step d) is

Proposition

Given σ^* a strategy that $*$approach C up to $\varepsilon > 0$ in the game G^*, there exists σ a strategy that approach C up to 2ε in the game G.

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