On Second-Order Optimality Conditions for Conic Programming

Héctor Ramírez C.[1]

[1] Departamento Ingeniería Matemática & Centro Modelamiento Matemático, Universidad de Chile, Santiago de Chile

6èmes Journées Franco-Chiliennes d'Optimisation
May 19th, 2008 Université du Sud Toulon-Var

Outline

1. Introduction and Motivation
   - Formulation of Our Problem
   - Applications of SDP and SOCP
   - Optimality Conditions
   - Constraint Qualification Conditions
   - Reduction Approach
2. Main results
   - Duality Results for Conic Programming
   - Strong Regularity Condition

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Consider the optimization problem over a closed convex cone K

$$\min_{x \in \mathbb{X}} f(x) \;;\; g(x) \in K \subseteq \mathbb{Y} \tag{P}$$

where $\mathbb{X}$ and $\mathbb{Y}$ are finite dimensional Hilbert spaces.

For instance:

$$K = \{0\} \times \mathbb{R}^m_- \subset \mathbb{R}^p \times \mathbb{R}^m, \tag{NLP}$$

$$K = S^m_- \tag{SDP}$$

or

$$K = Q_{m_1+1} \times Q_{m_2+1} \times \ldots \times Q_{m_J+1}, \tag{SOCP}$$

where $Q_{m_j+1} = \{y = (y_0, \bar{y}^\top) \in \mathbb{R} \times \mathbb{R}^{m_j} \;:\; y_0 \geq \|\bar{y}\|\}$

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Applications

- Minimization of Maximum Eigenvalue of $G(x)$

$$\min_{t\in\mathbb{R},\, x\in\mathbb{R}^n} t \;;\; G(x) - tI \preceq 0$$

- Robust Linear Programming

$$\min_{x\in\mathbb{R}^n} \{ f(x) = c^\top x \,;\, a_i^\top x \le b_i \quad \forall a_i \in \mathcal{E}_i,\; i = 1, \ldots, m \} \qquad \text{(RLP)}$$

where $P_i = P_i^\top \succeq 0$, and $\mathcal{E}_i := \{ \bar{a}_i + P_i u \;:\; \|u\| \le 1 \}$

The robust linear constraint can be reformulated as follows

$$\max\{ a_i^\top x \;:\; a_i \in \mathcal{E}_i \} = \bar{a}_i^\top x + \|P_i x\| \le b_i.$$

which is of the form: $\begin{pmatrix} b_i - \bar{a}_i^\top x \\ P_i x \end{pmatrix} \in \mathcal{Q}_{n+1}$

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First Order Optimality Conditions

Consider the optimization problem over a closed convex cone K

$$\min_x f(x) \; ; \; g(x) \in K \tag{P}$$

Karush-Kuhn-Tucker Conditions

We say that (x^*, y^*) is a KKT-point ($y^* \in \Lambda(x^*)$) if it satisfies

$$\begin{aligned} \nabla_x L(x^*, y^*) = \nabla f(x^*) + Dg(x^*)^\top y^* = 0, \\ y^* \in N_K(g(x^*)), \end{aligned} \tag{KKT}$$

where $N_K(z)$ is the normal cone to K at $z \in K$

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$$\begin{aligned} \nabla_x L(x^*, y^*) = \nabla f(x^*) + Dg(x^*)^\top y^* &= 0, \\ \langle g(x^*), y^* \rangle &= 0, \\ g(x^*) \in K,\ y^* \in K^- \ (= -K)&, \end{aligned} \tag{KKT}$$

where K^- is the negative polar cone of K

Let x^* be a feasible point of (P).

Definition

We say that x^* satisfies Robinson's const. qualif. cond. if

$$Dg(x^*)\mathbb{X} + T_K(g(x^*)) = \mathbb{Y} \tag{Rob}$$

where $T_K(g(x^*))$ is the tangent (or Bouligand) cone of K at $g(x^*)$

NLP case: Mangasarian-Fromovitz condition:
$\nabla g_i(x^*)$, for all $i \in \{1, ..., p\}$, are l.i. and $\exists h \in \mathbb{R}^n$ such that:

$$\nabla g_i(x^*)^\top h = 0, \quad \forall i \in \{1, ..., p\},$$
$$\nabla g_i(x^*)^\top h < 0, \quad \forall i \in \{p+1, ..., p+m\} \text{ s.t. } g_i(x^*) = 0$$

SDP case: $\exists\, h \in \mathbb{R}^n$ such that $E^\top Dg(x^*) hE \prec 0$,
where the columns of E are an orthonormal basis of $\operatorname{Ker} g(x^*)$.

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$$Dg(x^*)\mathbb{X} + \mathrm{lin}(T_K(g(x^*))) = \mathbb{Y}, \tag{NDG}$$

where $\mathrm{lin}(C)$ is the biggest linear space contained in C

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SDP case: Either $g(x^*) \in \mathrm{int}\, K$ or $h \in \mathbb{R}^n \to \psi(h) = E^\top Dg(x^*)hE$ is onto, where the columns of E are an orthonormal basis of $\mathrm{Ker}\, g(x^*)$

Also called Transversality condition in Shapiro '96

Constraint Qualification Conditions

Nondegeneracy Condition

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Remarks about Nondegeneracy Condition

- (NDG) implies (Rob) ($\Rightarrow \exists\, y^*$ (KKT)-multiplier)
- (NDG) implies a unique y^*
- (NDG) $\Leftrightarrow \exists!\, y^*$, when strict complementarity condition holds:

$$y^* \in \mathrm{ri}\, N_K(g(x^*))$$

Bonnans & Shapiro 2000

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Reduction Approach

Let $K \subseteq \mathbb{X}$ and $\hat{K} \subseteq \mathbb{Y}$ be closed, convex cones

Definition

K is said to be (pointed, C^2 and cone) reducible at $s^* \in K$ to $\hat{K}$ if there exist a neighborhood N of s^* and a C^2 mapping $\Xi : N \to \mathbb{Y}$ such that

- $\Xi(s^*) = 0$ *and* $T_{\hat{K}}(\Xi(s^*))$ *is a pointed cone*
- *for all* $s \in N$, $s \in K$ *iff* $\Xi(s) \in \hat{K}$
- $D\Xi(s^*) : \mathbb{X} \to \mathbb{Y}$ *is onto*

When K is reducible for all $s^* \in K$ to some $\hat{K}$ (possibly depending on s^*), we simply say that K is reducible

In this talk K is supposed to be reducible

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Examples:

SDP case: S_+^m is reducible. Indeed, for every matrix $\hat{Z} \in S_+^m$:
(i) If $\hat{Z} \succ 0$, take $\hat{K} = \{0\}$ and $\Xi(Z) = 0$, (ii) Else, take $\hat{K} = S_+^k$ and $\Xi(Z) = E^\top ZE$, where columns of $E \in \mathbb{R}^{m \times k}$ are an orth. basis of $\operatorname{Ker} \hat{Z}$

Proposition

Let x^* be a feasible point of problem (P).
Hence, nondegeneracy of x^* is equivalent to saying that the mapping $h \to D\mathcal{A}(x^*)h$ is onto, where $\mathcal{A} := \Xi \circ g$

Bonnans & Shapiro 2000

Examples:

SOCP case: Q_{m+1} is reducible. Indeed, for every vector $\hat{s} \in Q_{m+1}$: (i) If $\hat{s} = 0$, take $\hat{K} = Q_{m+1}$ and $\Xi(s) = s$, (ii) If $\hat{s}_0 > \|\bar{\hat{s}}\|$, take $\hat{K} = \{0\}$ and $\Xi(s) = 0$, (iii) If $0 \neq \bar{s}_0 = \|\bar{s}\|$, take $\hat{K} = \mathbb{R}_-$ and $\Xi(s) = \|\bar{s}\| - s_0$.

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Second-Order Optimality Conditions

For the problem (P), the SONC at x^* is given by

$$\sup_{y\in\Lambda(x^*)} \left\{ D^2_{xx}L(x^*,y)(h,h) - \sigma(y,\mathcal{T}(h)) \right\} \geq 0, \ \ \forall\, h \in C(x^*),$$

where

$$C(x^*) = \{h \in \mathbb{X} \,:\, Dg(x^*)h \in T_K(g(x^*)),\ \nabla f(x^*)^\top h = 0\},$$

$\sigma(y,K) := \sup_{w\in K}\langle w, y\rangle$ ($\sigma(y,\mathcal{T}(h))$ is known as the sigma term) , and

$$\mathcal{T}(h) := T^2_K(z,d) = \left\{ w \in \mathbb{Y} : \mathrm{dist}(z + td + \frac{1}{2}t^2 w, K) = o(t^2),\ t \geq 0 \right\}$$

is the second order tangent set to K at $z = g(x^*)$ in the direction $d = Dg(x^*)h$

Bonnans, Cominetti & Shapiro '99

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For the problem (P), the SOSC at x^* is given by

$$\sup_{y\in\Lambda(x^*)} \left\{ D^2_{xx}L(x^*,y)(h,h) - \sigma(y,\mathcal{T}(h)) \right\} > 0, \quad \forall\, h \in C(x^*) \setminus \{0\},$$

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is the second order tangent set to K at $z = g(x^*)$ in the direction $d = Dg(x^*)h$

Bonnans, Cominetti & Shapiro '99

Second-Order Optimality Conditions

For the problem (P), the SOSC at x^* is given by

$$\sup_{y \in \Lambda(x^*)} \left\{ D^2_{xx} L(x^*, y)(h,h) - \sigma(y, \mathcal{T}(h)) \right\} > 0, \quad \forall\, h \in C(x^*) \setminus \{0\},$$

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Strong Second-Order Sufficient Optimality Conditions

For the problem (P), the SSOSC at x^* and $y^* \in \Lambda(x^*)$ is given by

$$D^2_{xx}L(x^*, y^*)(h, h) - \xi_*(h) > 0, \quad \forall\, h \in \mathrm{Sp}(C(x^*)) \setminus \{0\}$$

where $\mathrm{Sp}(C) := \mathbb{R}_+(C - C)$ is the linear space generated by C, and $\xi_*(h)$ is a quadratic function on h, that depends on y^*, $g(x^*)$ and on the reduction $\Xi_{g(x^*)}$

Remark: When $Dg(x^*)h \in T_K(g(x^*))$, it holds $\mathcal{T}(h) \neq \emptyset$, obtaining

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For the problem (SDP), this sufficient condition looks like follows:

$$h^\top \nabla^2_{xx} L(x^*, y^*)h + h^\top \mathcal{H}(x^*, y^*)h > 0, \ \forall h \in \mathrm{Sp}(C(x^*)) \setminus \{0\},$$

where $\mathcal{H}(x^*, Y^*)_{ij} := -2Y^* \cdot [D_{x_i} g(x^*) g(x^*)^\dagger D_{x_j} g(x^*)]$

For the problem (SOCP), this sufficient condition looks like follows:

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Second-Order Optimality Conditions

Why should we study second-order optimality conditions involving an additional term (e.g. σ-term)?

- It seems to be the "right" conditions for studying convergence of algorithms for solving non-linear conic problems.

  For instance, S-SDP methods: Fares et al.'02, Correa & Ramírez'04, Garcés, Gómez & Jarre'08

- It allows us to understand some subtle differences between NLP and other conic programming, such as SDP and SOCP.

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Outline

Extended Wolfe's Dual

Definition

We define the extended Wolfe's dual as follows

$$
\begin{aligned}
\max_{x\in\mathbb{X},\, y\in\mathbb{Y}} & \; L(x,y) = f(x) + \langle y, g(x)\rangle \\
\text{subject to } & \; \nabla_x L(x,y) = 0, \\
& \; y \in K^-(= -K),
\end{aligned}
\tag{EWD}
$$

where K^- is the negative polar cone of K

Dual Strong Second-Order Sufficient Condition

For simplicity, from now on we suppose that K is self-polar ($K^- = -K$)

Let (x^*, y^*) be KKT-point of problem (P)

Definition

We say that the Dual Strong Second-Order Sufficient Condition (DSSOSC) holds at (x^*, y^*) if $\nabla^2_{xx}L(x^*, y^*)$ is nonsingular and

$$w^\top[\nabla^2_{xx}L(x^*, y^*)]^{-1}w - \xi^D_*(u) > 0, \ \forall (w, u) \in \mathbb{X} \times \mathbb{Y} \setminus \{0\}\,;$$
$$w = Dg(x^*)^\top u, \ u \in \mathrm{Sp}(T_{K^-}(y^*) \cap g(x^*)^\perp)$$

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DSSOSC and NDG

Proposition (Ramírez '08)

Let (x^*, y^*) be KKT-point of problem (P). Then:

- *$(x^*, y^*, 0, g(x^*))$ is a KKT-point for problem (EWD), that is, $(0, g(x^*)) \in \mathbb{R}^n \times K$ is a KKT-multiplier of (x^*, y^*) for problem (EWD)*
- *DSSOSC holds at (x^*, y^*) iff SSOSC holds at $(x^*, y^*, 0, g(x^*))$ for problem (EWD)*

Theorem (Ramírez '08)

Let (x^*, y^*) be a KKT-point of (P). Then, if DSSOSC holds at (x^*, y^*) we obtain that x^* is nondegenerate for problem (P)

These results extend similar ones obtained for NLP (Mangasarian '84)

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Strongly Regular Solutions

$$(KKT) \quad \Leftrightarrow \quad (z=(x,y)) \qquad 0 \in F(z)+N(z) \tag{GE}$$

We linearize (GE) at $z^* = (x^*, y^*)$ and parameterize by δ:

$$\delta \in F(z^*) + DF(z^*)(z - z^*) + N(z) \tag{LE_δ}$$

Definition (Robinson '80)

z^* is called a strongly regular (SR) solution of (GE) if there exists a neighborhood $\mathcal{N}$ of z^* s.t. for all δ small enough, (LE_δ) has a unique solution $z^*(\delta)$ in $\mathcal{N}$, which is Lipschitz on δ

This definition can be applied to:

$$K = S^m_- \text{ (SDP)} \qquad \text{and} \qquad K = Q_{m+1} \text{ (SOCP)}$$

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Remarks about Strong Regularity

- Existence of SR solution implies (Rob)
- If K is reducible, Existence of SR solution implies (NDG)
- SR “extends” the *Implicite Function Theorem*
  See Robinson ’80
- SR stable under small perturbations
- Characterization of SR in nonlinear programming
  Bonnans & Sulem’95 / Dontchev & Rockafellar’98
- It is still an open problem for a general conic optimization framework

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SDP case:

Theorem (Bonnans & Ramírez '05, Sun '06)

Let x^* be a sol. of (SDP) and Y^* its corresponding (KKT)-multiplier. Then,

If (x^*, Y^*) is a strongly regular solution iff
x^* is nondegenerate and the following strong SOSC holds:

$$h^\top \nabla^2_{xx} L(x^*, Y^*)h + h^\top H(x^*, Y^*)h > 0, \ \forall h \in \mathrm{Sp}(C(x^*)) \setminus \{0\},$$

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Characterization of the Strong regularity

SOCP case:

Theorem (Bonnans & Ramírez '05)

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