An extreme eigenvalue problem for a two-phase conductor[1]

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Plan

- Problem Statement.
- Existence-Difficulties.
- Symmetry and Existence.
- Improvements.
- Numerical Experiments.

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Problem Statement

Eigenvalue Problem for Conductors

- $\Omega \subset \mathbb{R}^n$ - design region.
- $0 < \alpha < \beta$ - conductivity coefficients.
- $\omega \subset \Omega$ - region occupied by β.
- $$\lambda^1(\omega) := \min_{u \in H_0^1(\Omega)} \frac{\int_\Omega (\alpha \chi_{\Omega \setminus \omega} + \beta \chi_\omega) |\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

Optimization Problem.

- m-constant, $0 < m < |\Omega|$.
- $$\inf \left\{ \lambda^1(\omega) : \omega \subset \Omega, \omega \text{ measurable}, |\omega| = m \right\}.$$

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Questions of Interest

- Does there exist a minimizer for the problem?
- How does it look like? - To obtain characterizations of minimizers.

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Existence

General Formulation

$$\inf \{F(\omega) : \omega \in \mathcal{A}\} \ .$$

$\mathcal{A}$- admissible shapes.

Weierstrass-Tonnelli Existence Theorem

If we can give a topology on $\mathcal{A}$ for which

1. F is lower-semicontinuous and,
2. the level sets of F in $\mathcal{A}$ are compact

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Existence-Difficulties

Finding a topology which serves.

Haussdorff convergence of sets

$$\omega_n \xrightarrow{H} \omega \text{ if } d_H(\omega_n, \omega) \to 0 \, ,$$

where

$$d_H(\omega_n, \omega) = \max \left\{ \sup_{x \in \omega_n} d(x, \omega), \sup_{x \in \omega} d(x, \omega_n) \right\} \, ,$$

- $\omega \mapsto \lambda_1(\omega)$ is continuous but,
- $\{\omega : \omega \subset \Omega, \omega$ measurable $, |\omega| = m\}$ is not compact.

Supplementary constraints

Perimeter constraint, convex inclusions, number of connected components, capacity conditions etc...make the constraint set compact for the above topology cf. Bucur and Buttazzo, Henrot and Pierre.

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Change of Perspective

$$\mu_1(\nu) := \lambda_1(\omega) \text{ if } \nu = \alpha\chi_{\Omega\setminus\omega} + \beta\chi_\omega\,, |\omega| = m$$

Search for a Topology

Admissible set -$\mathcal{C} := \{\nu : \nu = \alpha\chi_{\Omega\setminus\omega} + \beta\chi_\omega\,, \omega \subset \Omega\,, |\omega| = m\}$

- Any topology on $\mathcal{C}$ which gives pointwise a. e. convergence of ν *a priori* renders it non-compact.
- $\mathcal{C}$ relatively compact in $L^\infty(\Omega)$ for weak-$*$ topology but μ_1 not lower semi-continuous.

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Existence-Difficulties ...continued

Options-Relaxation

- Enlarge the solution space. Take $\mathcal{C} := \left\{ \nu \in L^\infty(\Omega) : \alpha \leq \nu\beta \, , \int_\Omega \nu(x)\, dx = \alpha(|\Omega| - m) + \beta m \right\}$
- To find the lower-semicontinuous envelope of the functional μ_1 on $\mathcal{C}$ for the weak-$*$ convergence on $L^\infty(\Omega)$.
- Matrix formulation of the coefficients and a different notion of matrix convergence (G- convergence)due to Spagnolo, Murat-Tartar is involved in this description.
- Solutions in this framework - show microstructure- studied by Cox-Lipton ARMA '96.

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Classical Solutions-Existence

Very few results are available.

One Dimension

- Kreĭn in 1955.
- Uses the equivalence with the first eigenvalue problem for vibrating strings.

$$\mu_1(\rho) = \min_{u \in H_0^1(\Omega)} \frac{\int_0^L |\nabla u|^2(y)dy}{\int_0^L \rho(y)|u|^2(y)dy}$$

where $\rho(y) = \nu(T^{-1}(y))$ and $T : [0,1] \to [0,L]$ with $T(x) = \int_0^x \frac{1}{\nu(s)}\, ds$.

- ρ satisfies similar constraints. μ_1 is continuous for weak-$*$ convergence.
- Precise minimizer consists in taking β in the middle. Shown by symmetrization.

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Higher dimensions ?

Membrane Problem

- Have existence for the "vibrating membrane problem" in any dimension cf. Cox and McLaughlin *Appl. Math. Optimization* '90.
- In a ball, the solution has the same structure as in one-dimension.
- In a symmetric domain one has symmetric minimizers. By Symmetrization.

Conduction $\longleftrightarrow$ Membrane?

Is there a transformation which gives an equivalence between the eigenvalue problems for conduction and membranes in dimensions ≥ 2?

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Theorem (Alvino, Lions, Trombetti *Nonlin. Anal. '89*)

There exists a classical symmetric minimizer.

Proof

Requires a fine symmetrization result.

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Schwarz Symmetrization

Definition

- $\Omega = B(0,1)$, $u : \Omega \to \mathbb{R}^+$ bounded.
- $\Omega_c = \{x \in \Omega : f(x) \geq c\}$, $\Omega_c^* = B(0, r_c)\,, |\Omega_c^*| = |\Omega_c|$.
- $f^*(x) := \sup\{c : x \in \Omega_c^*\}$.

- (Equimeasurability) $|\{f \geq c\}| = |\{f^* \geq c\}|$.
- (Isoperimetric inequality) $P(\{f \geq c\}) \geq P(\{f^* \geq c\})$.

Consequences

- $\int_\Omega h(f(x))\,dx = \int_\Omega h(f^*(x))\,dx$. In particular for $h(s) = s^2$.
- (Hardy-Littlewood Inequlaity) $\int_\Omega f(x)g(x)\,dx \leq \int_\Omega f^*(x)g^*(x)\,dx$.
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Solution Strategy?

Extract minimizing sequences having symmetry.

Membrane Problem

- ρ_n minimizing sequence $\Rightarrow$ ρ_n^* another minimizing sequence

$$\mu_1(\rho_n) = \frac{\int_\Omega |\nabla u_n|^2(y)dy}{\int_\Omega \rho_n(y)|u_n|^2(y)dy} \geq \frac{\int_\Omega |\nabla u_n^*|^2(y)dy}{\int_\Omega \rho_n^*(y)|u_n^*|^2(y)dy} \geq \mu_1(\rho_n)$$

Conduction Problem

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- A fine result proved using concentration compactness. Would require dexterity to obtain this for other kinds of symmetrizations.

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- ν_n minimizing sequence $\not\Rightarrow$ ν_n^* another minimizing sequence

Theorem (Alvino, Lions and Trombetti)

Given ν and u, there exists $\tilde{\nu}$ radially symmetric with $\nu^* = (\tilde{\nu})^*$ such that

$$\int_\Omega \nu |\nabla u|^2(x)\,dx \geq \int_\Omega \tilde{\nu} |\nabla u^*|^2(x)\,dx$$

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$$\int_\Omega \nu \left|\nabla u\right|^2(x)\,dx \geq \int_\Omega \widetilde{\nu}\left|\nabla u^*\right|^2(x)\,dx$$

- A fine result proved using concentration compactness. Would require dexterity to obtain this for other kinds of symmetrizations.

Existence

Reduction

$$\inf\{\lambda_1(\nu) : \nu \in \mathcal{C}\} = \inf\{\lambda_1(\nu) : \nu \in \mathcal{C}^s\}$$

First Existence Result

Existence in

$$\mathcal{K}^s := \left\{\nu : \exists \nu_n \in \mathcal{C}^s, \nu_n^{-1} \overset{*}{\rightharpoonup} \nu^{-1}\right\}$$

as $\lambda_1 \llcorner \mathcal{K}^s$ is continuous for $\nu_n \xrightarrow{r} \nu \iff \nu_n^{-1} \overset{*}{\rightharpoonup} \nu^{-1}$.

Classical Existence

$J : \nu^{-1} \mapsto \left(\lambda^1(\nu)\right)^{-1}$ is a convex map on the convex set $\left\{\nu^{-1} : \nu \in \mathcal{K}^s\right\}$. There is always an extremum point which maximizes a convex function on a convex set.

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Improvements

First existence result is in an enlarged set; Alvino, Lions and Trombetti theorem may not extend to other symmetric domains.

Lemma-Alvino and Trombetti

Given ν and u, there exists $\widetilde{\nu} \in \mathcal{K}^s$ such that

$$\int_\Omega \nu \left|\nabla u\right|^2(x)\,dx \geq \int_\Omega \widetilde{\nu} \left|\nabla u^*\right|^2(x)\,dx$$

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$$\inf\{\lambda_1(\nu) : \nu \in \mathcal{C}\} = \min\{\lambda_1(\nu) : \nu \in \mathcal{K}^s\}$$

Observations

- Proof of Alvino and Trombetti Lemma uses only the co-area formula, the properties of symmetrization and the isoperimetric inequality.
- We give a refined proof. Possible to change Schwarz symmetrization for Steiner symmetrization. $\Rightarrow$ existence of a symmetric minimizer.
- Existence of a classical minimizer? uniqueness? exact shape? etc..

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