Optimization of trusses under uncertain loads.

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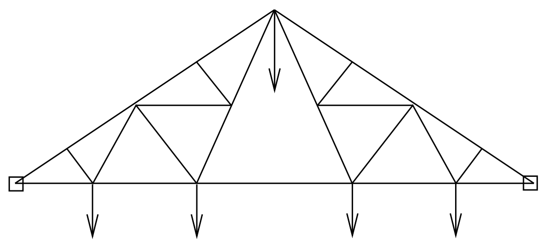
- **Optimization of trusses**
  - Standard minimum compliance truss design.
  - Instabilities and the standard multiload model.
  - **Random Perturbations: Minimizing the expected compliance.**
  - Numerical examples.
  - **Random Perturbations: Minimizing the variance.**
  - Numerical examples.

Optimization of trusses

Introduction

- Definition.
- Problem: find the **best structure** able to carry a external nodal force.
- Constraints: mechanical equilibrium , total volume and others . . .
- Minimize the compliance.



Minimum compliance truss design

$$(\mathcal{D}) \qquad \begin{aligned} \min_{\lambda, u} \ & \frac{1}{2} f^T u \\ s.t. \ & K(\lambda) u = f \\ & \lambda \in \Delta_m \end{aligned}$$

$\frac{1}{2} f^T u$ is called *Compliance*.

Mathematical formulation

- $\lambda \in \mathbb{R}^m$; m number of bars.
- n grades of freedom.
- $f \in \mathbb{R}^n$ nodal load vector.
- $u \in \mathbb{R}^n$ nodal displacements.

$$\boxed{\begin{aligned} &\min_{\lambda,u} \frac{1}{2} f^T u \\ &s.t.\ K(\lambda)u = f \\ &\qquad \lambda \in \Delta_m \end{aligned}}$$

- $\Delta_m = \{\lambda \in \mathbb{R}^m \mid \lambda \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1\}$.
- $K(\lambda) \in \mathbb{R}^{n \times n}$, *stiffness matrix*.

Stiffness Matrix

$$K(\lambda) = \sum_{i=1}^{m} \lambda_i K_i$$

where

$$K_i = b_i b_i^T \in \mathbb{R}^{n \times n}$$

K_i is dyadic.

Stiffness Matrix

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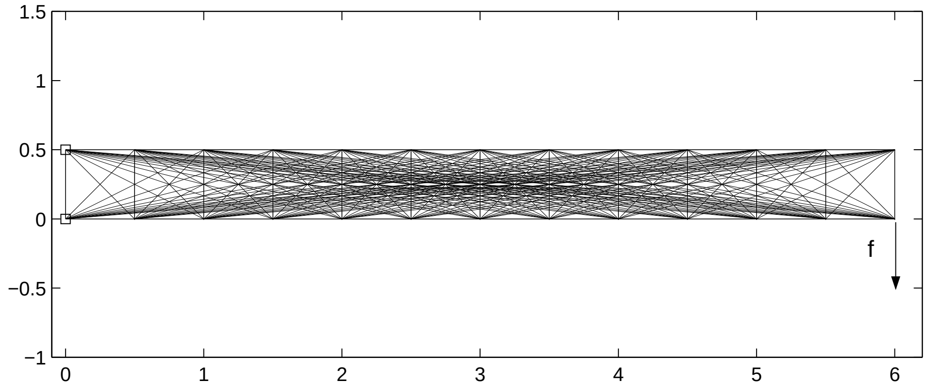
K_i is dyadic.

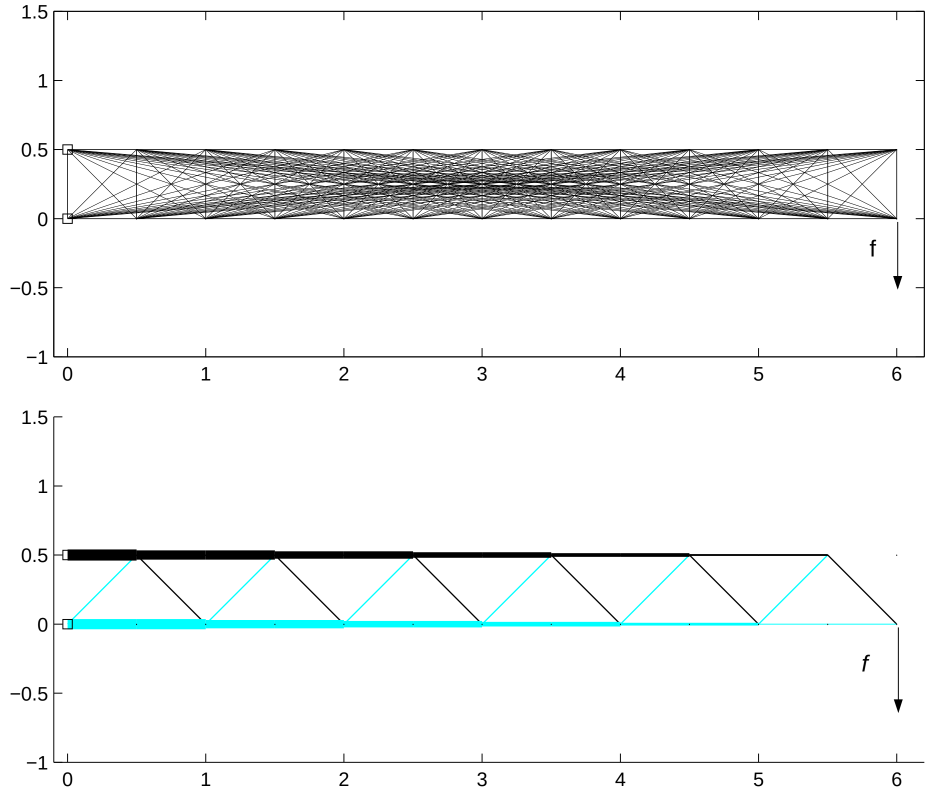
$$b_i = \frac{\sqrt{E_i}}{l_i} \gamma_i \in \mathbb{R}^n$$

- E_i: Young modulus.
- l_i: length of bar i.
- γ_i: *cosines/sines* vector.

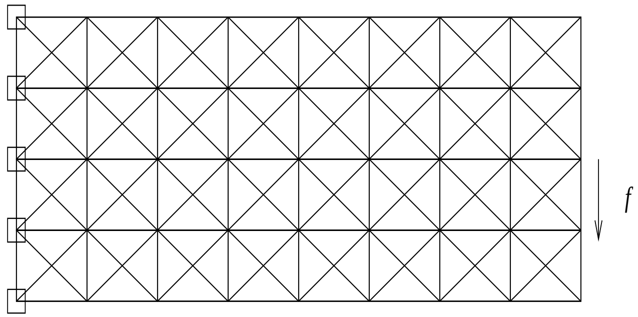
Ground structure approach



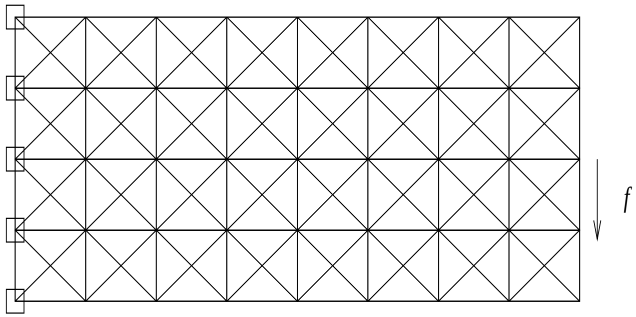
Ground structure approach

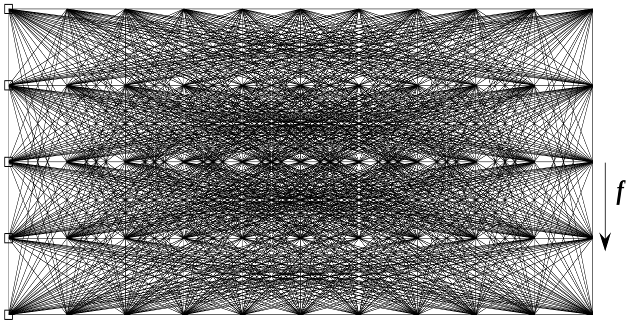


Ground structure approach



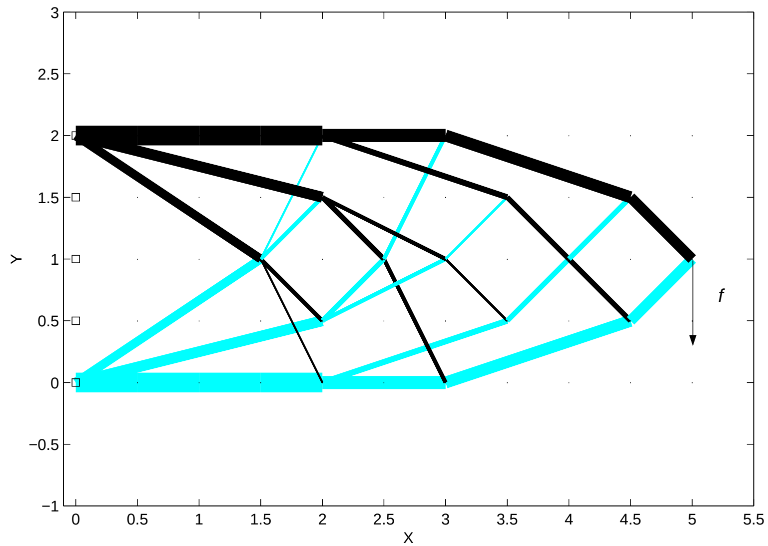
Ground structure approach





- Nodal positions are fixed in the reference configuration.
- We use a mesh full of nodes and bars.

Ground structure approach



Typically a large number of bar vanish at the optimum.

$(\mathcal{D})$

$$\min_{\lambda\in\mathbb{R}^m, u\in\mathbb{R}^n} \frac{1}{2}f^T u$$

$$s.t.\ K(\lambda)u = f$$

$$\lambda \in \Delta_m$$

$\frac{1}{2}f^T u$ is independent of u satisfying $K(\lambda)u = f$.

Minimax Formulation

$$(\mathcal{D}) \qquad \min_{\lambda\in\Delta_m}\{\frac{1}{2}f^T u \mid \; K(\lambda)u = f\}$$

$$(\mathcal{D}) \qquad \min_{\lambda\in\Delta_m}\max_{x\in\mathbb{R}^n}\{f^T x - \frac{1}{2}x^T K(\lambda)x\}$$

$\Updownarrow$ (Minimax Theorem)

$$(\mathcal{P}) \qquad \min_{x\in\mathbb{R}^n}\max_{1\leq i\leq m}\{\frac{1}{2}x^T K_i x - f^T x\}$$

Instabilities and the standard multiload model

Example

The design problem $(\mathcal{D})$ may produce unsatisfactory results in respect to mechanical stability (see Ben-Tal and Nemirovsky 1997)

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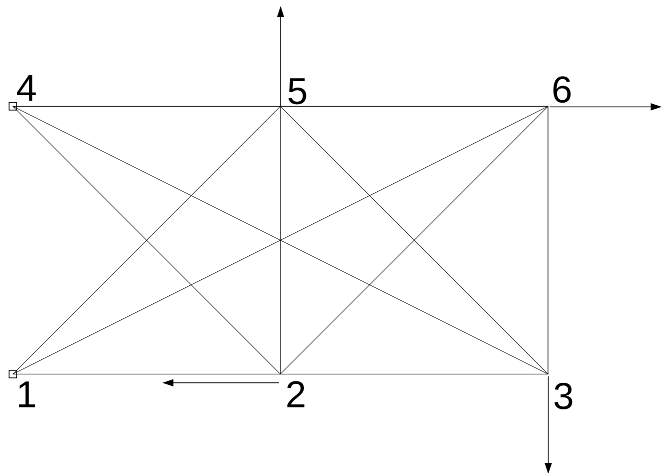

Example

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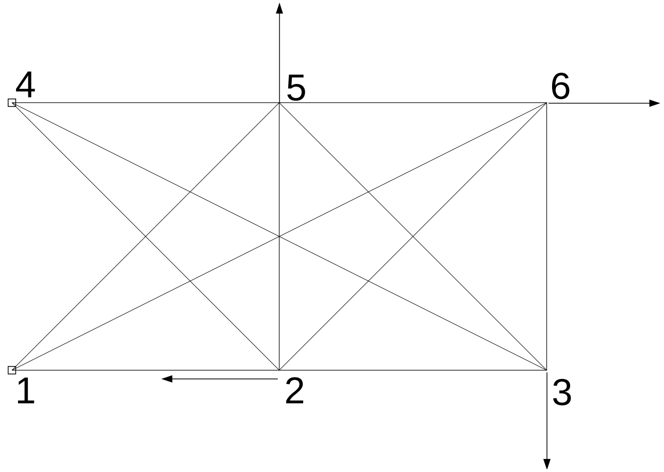
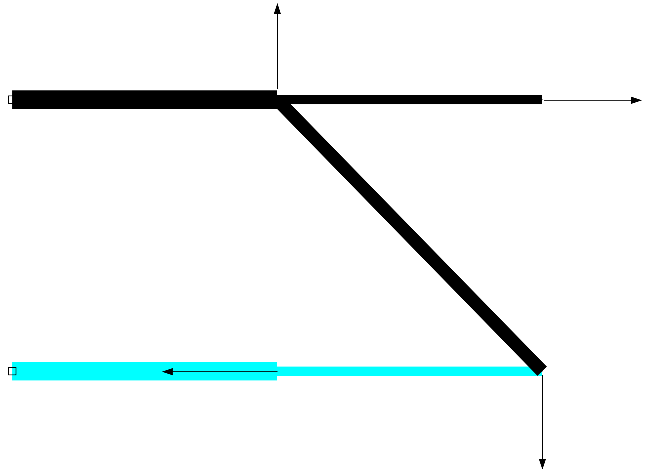




Multiload model

Standard multiload model

$$\min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^{k} \alpha_j f^{jT} u^j$$

$$s.t.\ K(\lambda) u^j = f^j, \quad j = 1, \ldots, k$$

$$\lambda \in \Delta_m$$

We minimize a weighted average of the compliances.

Defining

$$\hat{f} = (\alpha_1 f^{1T}, \ldots, \alpha_k f^{kT})^T \in \mathbb{R}^{n(k+1)},$$

$$\hat{K}_i = \mathrm{diag}(\alpha_1 K_i, \ldots, \alpha_k K_i) \in \mathbb{R}^{n(k+1)\times n(k+1)},$$

$$\hat{K}(\lambda) = \sum_{i=1}^{m} \lambda_i \hat{K}_i,$$

Then multiload model can be written as ($\mathcal{D}$).
remark: $\hat{K} \neq b_i b_i^T$.

Random Perturbations: Minimizing the expected compliance.

Random load model

Let $\xi \in \mathbb{R}^n$ be a perturbation on the load vector f.

$$\Psi(\xi,\lambda) = \begin{cases} \frac{1}{2}(f+\xi)^T x & \text{if } \lambda \in \Delta_m \text{ and } u \in \mathbb{R}^n \text{ such that} \\ & K(\lambda)u = f+\xi, \\ +\infty & \text{otherwise} \end{cases}$$

$$\Psi\colon (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \to (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R}))$$

Results to be proper, lower semicontinuous and convex.

Random load model

For each $\lambda \in \Delta_m$

$$\begin{array}{rccc} \Psi(\cdot,\lambda) : & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) & \longrightarrow & (\mathbb{R} \cup \{+\infty\}, \bar{\mathcal{B}}(\mathbb{R})) \\ & \xi & \longmapsto & \Psi(\xi,\lambda) \end{array}$$

is measurable.

Random load model

Minimum expected compliance design problem

$(\mathcal{D}; \mathbb{P})$ $$\min_{\lambda \in \Delta^m} \mathbb{E}_\xi[\Psi(\xi, \lambda)]$$

We assume that ξ is a random variable corresponding to an uncertain nodal load perturbation

$$\begin{array}{rccc} \xi : & (\Omega, \mathcal{A}) & \rightarrow & (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n)) \\ & \omega & \mapsto & \xi(\omega) \end{array}$$

Example: discrete perturbation

1. $\mathbb{P}$ with finite support $\mathrm{Sop}(\mathbb{P}) = \{\xi^1, \ldots, \xi^k\}$.

$$(\mathcal{D}; \mathbb{P}) \qquad \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} \sum_{j=1}^{k} \alpha_j f^{jT} u^j$$

$$s.t.\ K(\lambda) u^j = f^j, \quad i = 1, \ldots, k$$

$$\lambda \in \Delta_m$$

$(\mathcal{D}; \mathbb{P})$ is the *standard multiload model* where $\alpha_j = \mathbb{P}(\xi = \xi^j)$ and $f^j = f + \xi^j$, $j = 1, \ldots, k$.

Continuous case

Continuous case

Theorem 0.1. *Let $\xi : \Omega \to \mathbb{R}^n$ be a continuous random variable with mean vector $\mathbb{E}(\xi) = 0$ and Variance–covariance matrix $\mathrm{Var}(\xi) = PP^T$, with $P \in \mathbb{R}^{n\times k}$.*
Then the corresponding minimum expected compliance design problem $(\mathcal{D}; \mathbb{P})$ is given by

$$(\mathcal{D}; \mathbb{P}) \qquad \min_{\lambda\in\mathbb{R}^m} \; \underbrace{\frac{1}{2} f^T u}_{\text{mean load}} + \underbrace{\frac{1}{2}\mathrm{Trace}(P^T U)}_{\text{variance}}$$

$$s.t.\; K(\lambda)u = f$$

$$K(\lambda)U = P$$

$$\lambda \in \Delta_m$$

Continuous case

Remark $p^j,\ j = 1, \dots, k$, columns of P
Defining

$$\hat{f} = (f^T, p^{1T}, \dots, p^{kT})^T \in \mathbb{R}^{n(k+1)},$$

$$\hat{K}_i = \operatorname{diag}(K_i, K_i, \dots, K_i) \in \mathbb{R}^{n(k+1)\times n(k+1)},$$

$$\hat{K}(\lambda) = \sum_{i=1}^{m} \lambda_i \hat{K}_i,$$

Then $(\mathcal{D}; \mathbb{P})$ may be written as a multiload model.

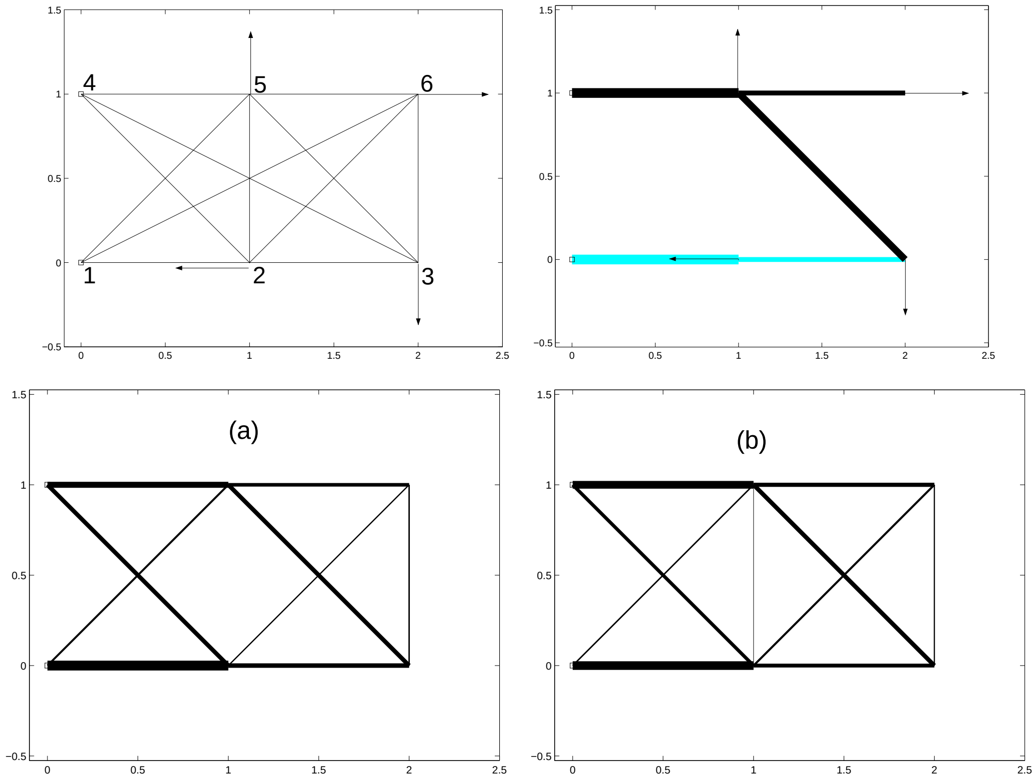
Random multidimensional independent perturbation

$\xi = \sum_{j=1}^{k} \varepsilon_j d^j$, where
$d^j \in \mathbb{R}^n$, $\mathbb{E}(\varepsilon_j) = 0$ and $\mathrm{Var}(\varepsilon_j) = \sigma_j^2$.

$$
\begin{aligned}
(\mathcal{D};\mathbb{P}) \qquad\qquad & \min_{\lambda \in \mathbb{R}^m} \frac{1}{2} f^T u + \frac{1}{2} \sum_{j=1}^{r} \sigma_j d^{jT} u^j \\
& s.t.\; K(\lambda) u = f \\
& \qquad K(\lambda) u^j = \sigma_j d^j \quad j = 1, \ldots, k \\
& \qquad \lambda \in \Delta_m.
\end{aligned}
$$

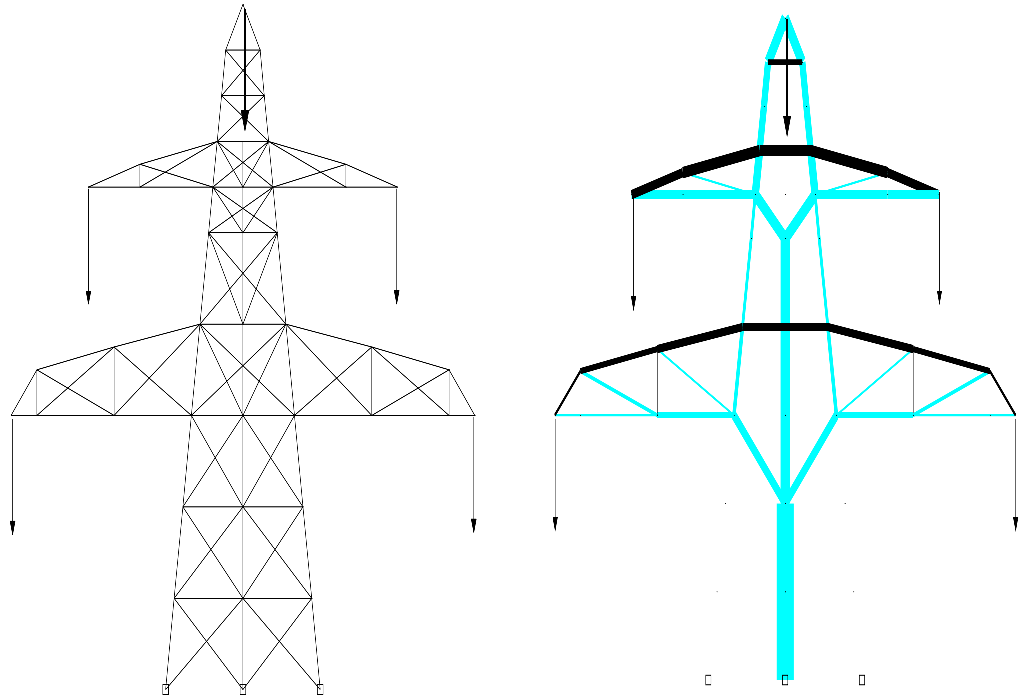
Numerical results

Toy example, Ben-Tal and Nemirowski 1997

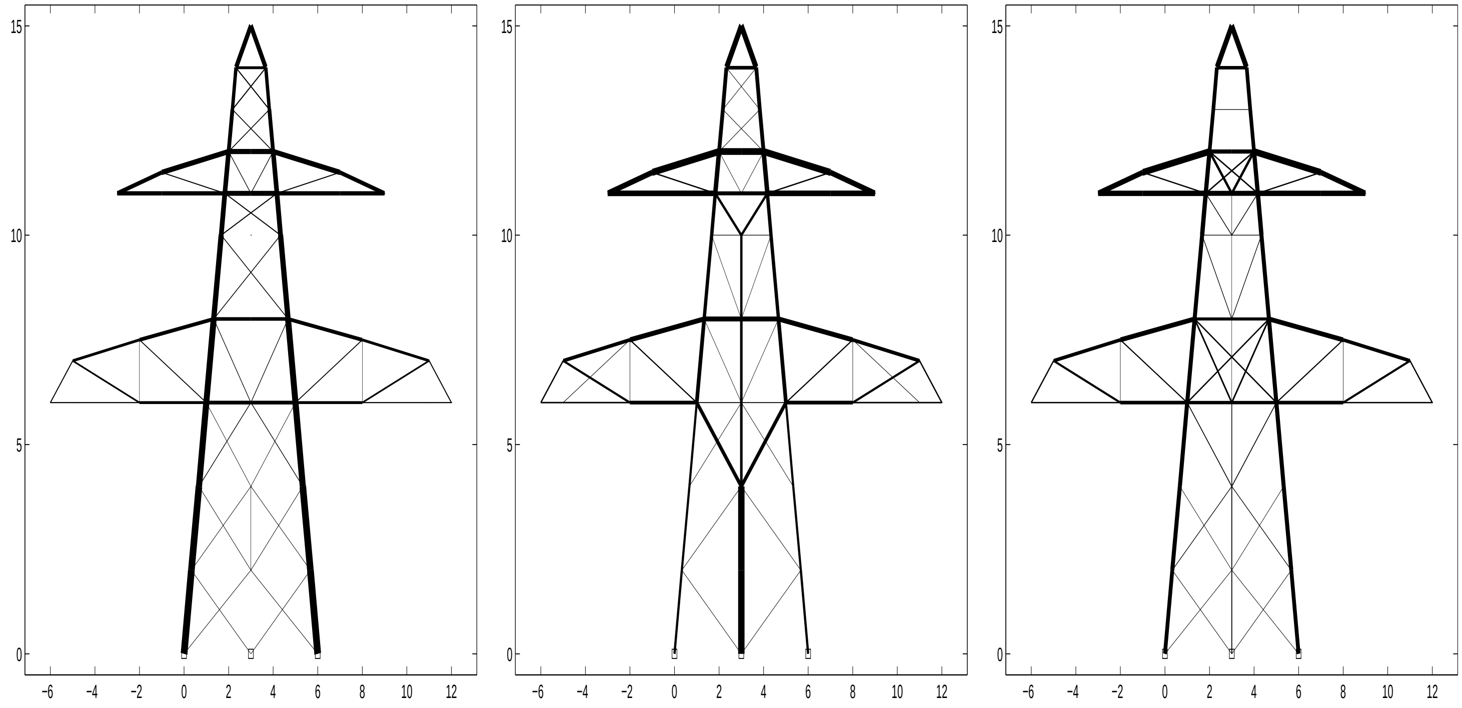


Problem	compl. mean	max compl.	min compl.
Standard single load	$\nexists$	$+\infty$	$+\infty$
(a)	0.025	0.053	0.013
(b)	0.023	0.051	0.012

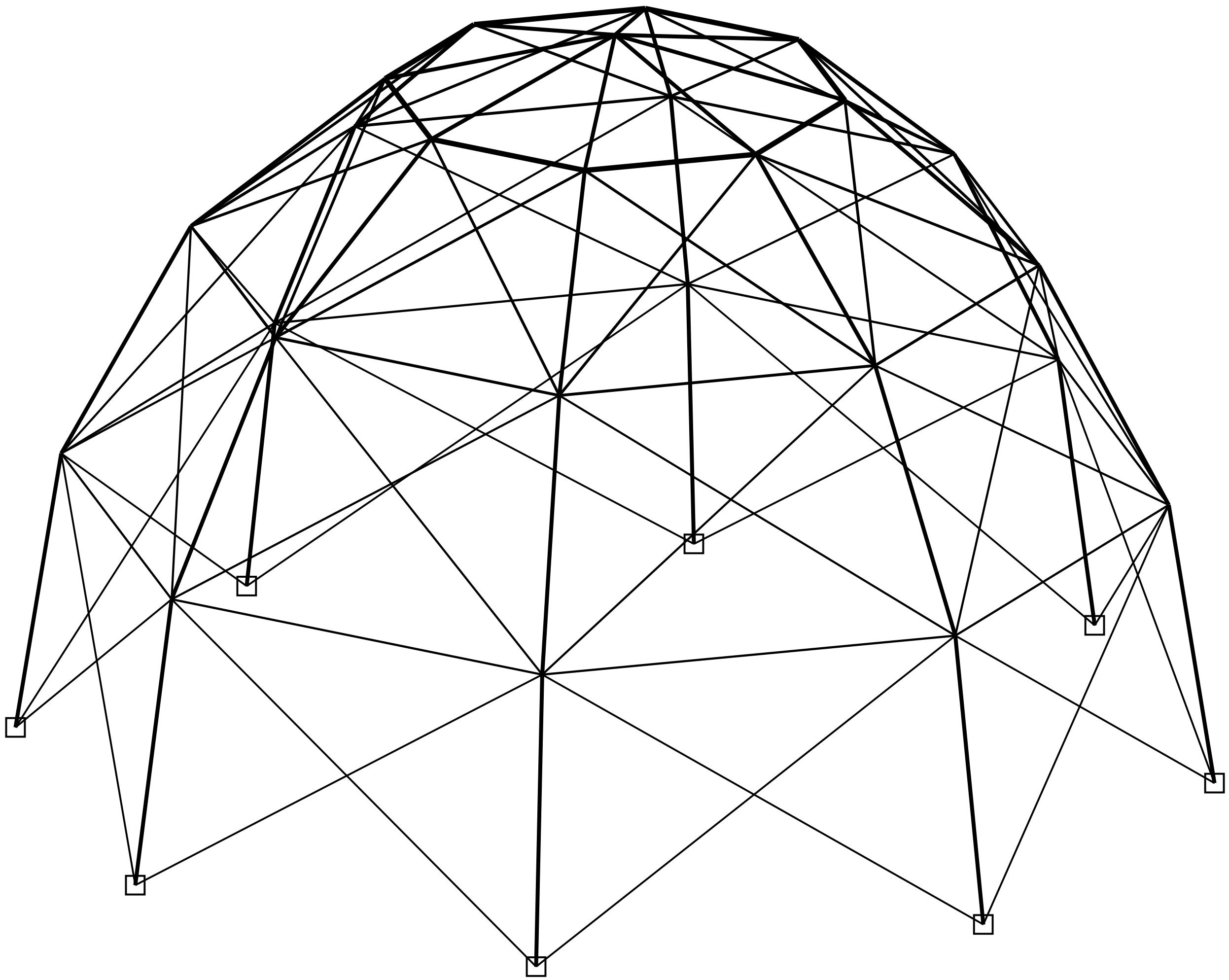
Electricity mast, Achtziger et al. 1992



Electricity mast



Dome



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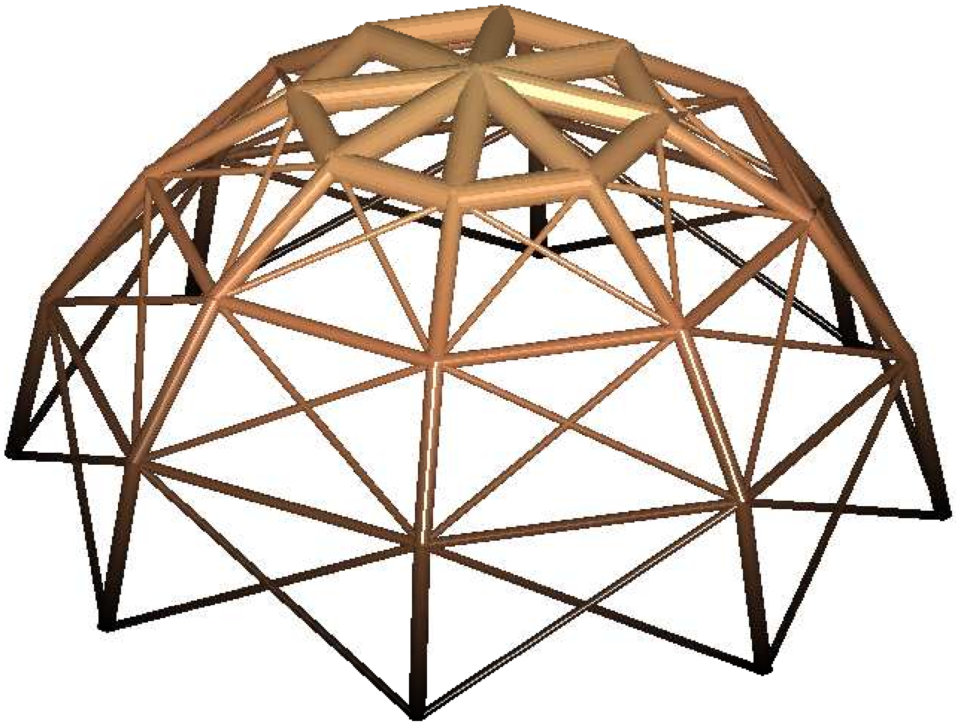


Random Perturbations: Minimizing the variance.

Stochastic model including variance

We consider

$$\min_{\lambda \in \mathbb{R}^m} \alpha \mathbb{E}_\xi[\Psi(\xi, \lambda)] + \beta \operatorname{Var}[\Psi(\xi, \lambda)].$$

Stochastic model including variance

We consider

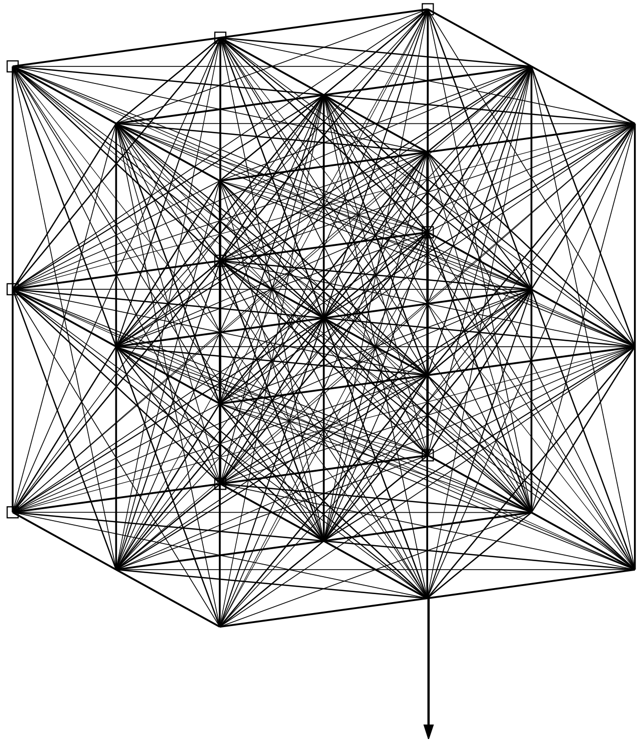
$$\min_{\lambda \in \mathbb{R}^m} \alpha \mathbb{E}_\xi[\Psi(\xi,\lambda)] + \beta \,\mathrm{Var}[\Psi(\xi,\lambda)]$$

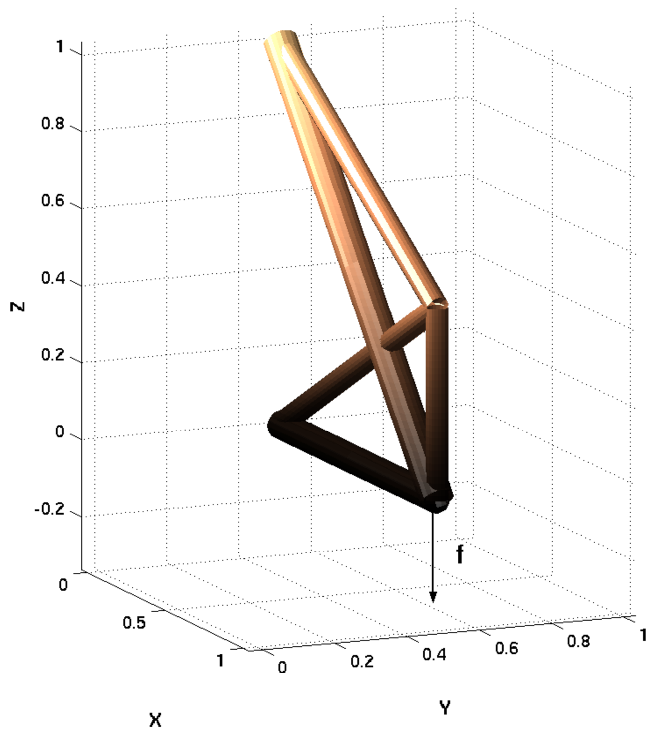
Theorem 0.2. $\xi \sim \mathcal{N}(0, PP^T)$. *Taking for simplicity* $\alpha = 0$ *and* $\beta = 1$.

$$\begin{aligned}
(\mathcal{D}; \mathbb{P}_{\mathit{var}}) \;\; \min_{\lambda \in \mathbb{R}^m} & \frac{1}{2}\left(\mathrm{Trace}((P^T U)^2) + 2 f^T U U^T f\right) \\
s.t.\; & K(\lambda) u = f \\
& K(\lambda) U = P \\
& \lambda \in \Delta_m
\end{aligned}$$

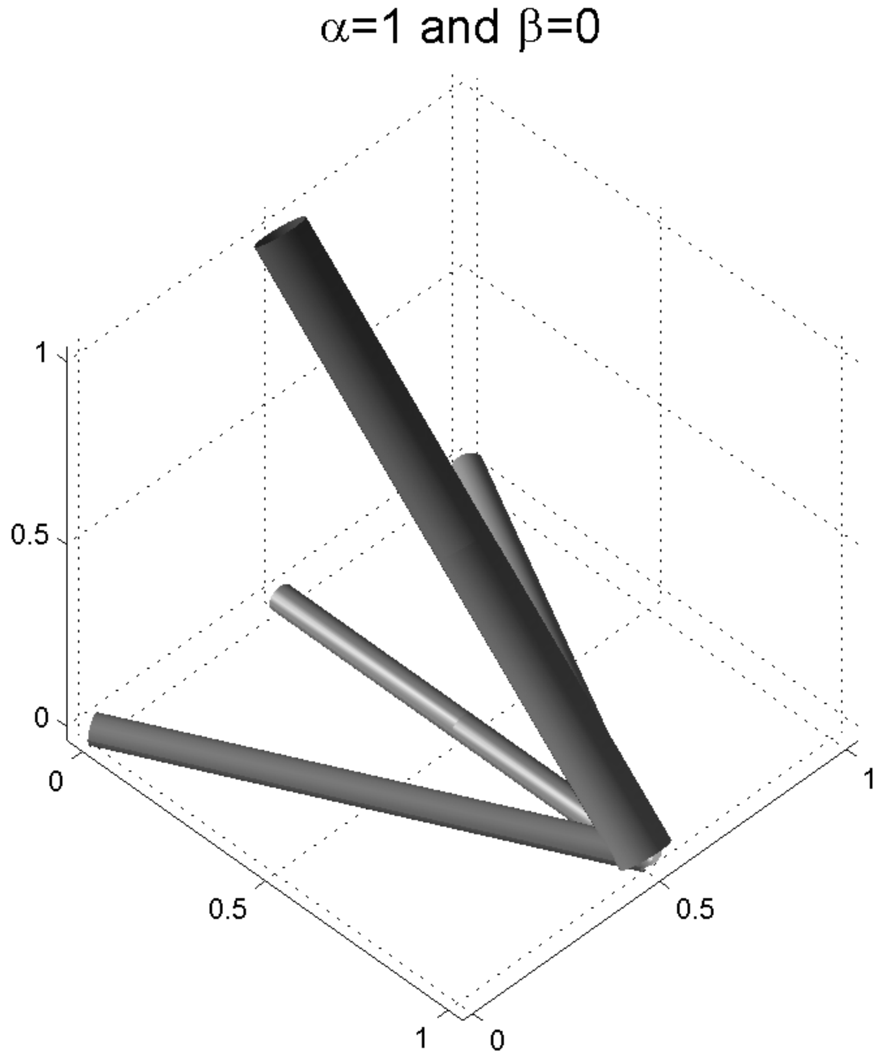
Apparent harder to solve.

Michel 3D

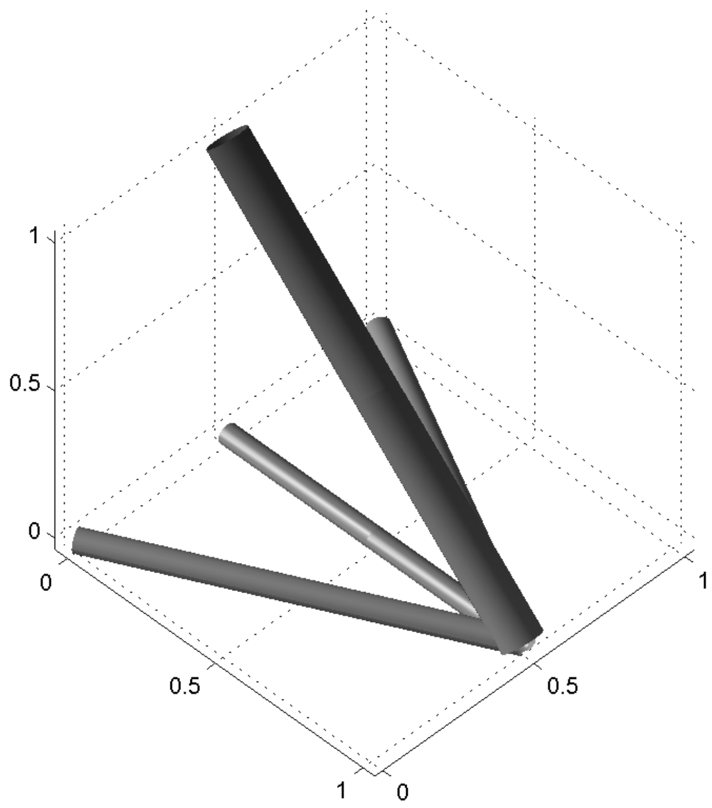




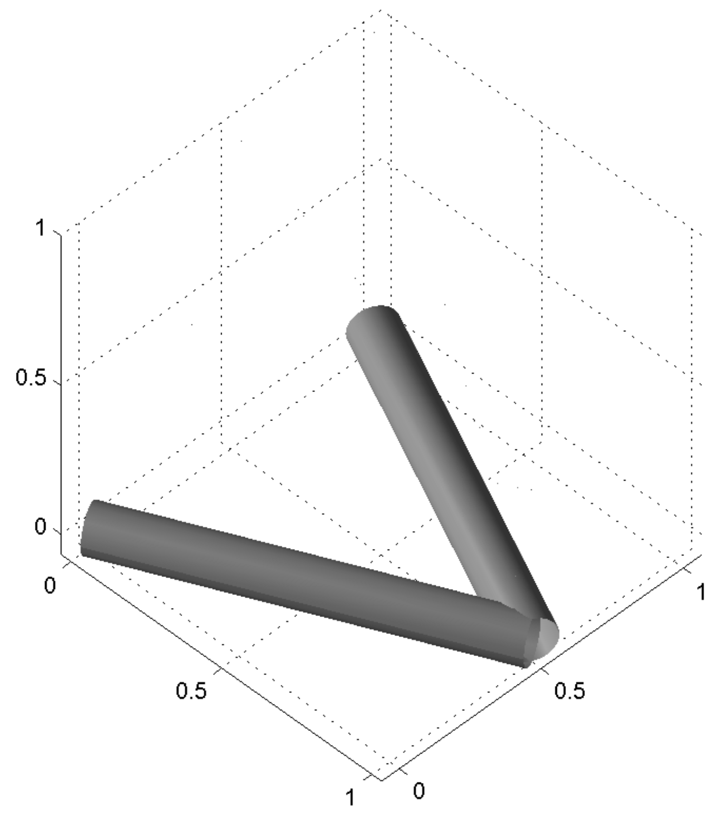
Michel 3D

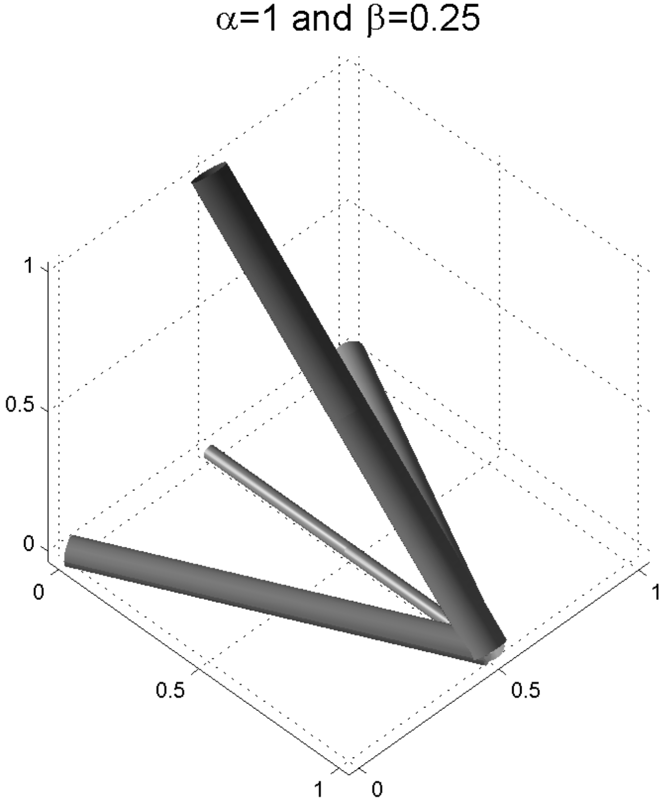


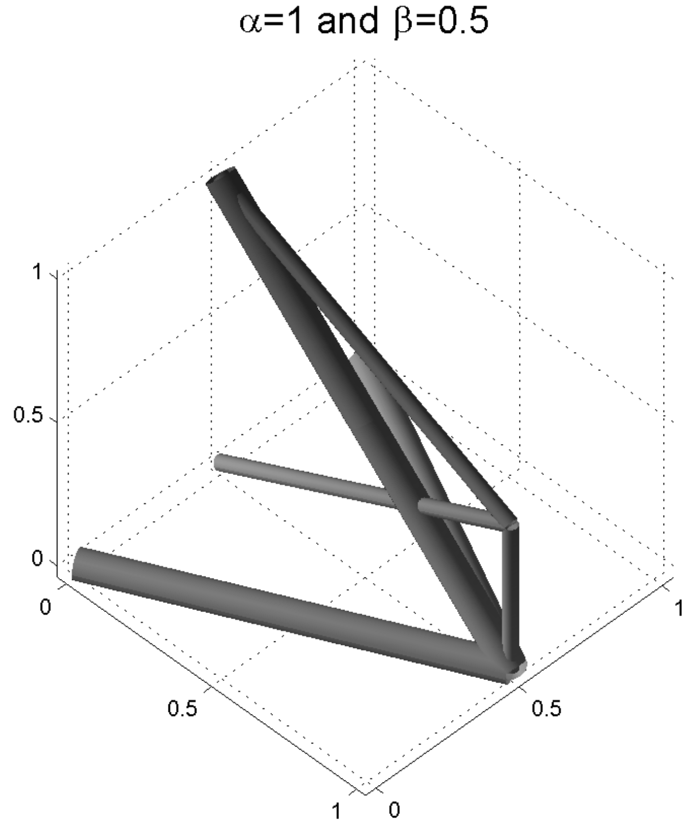
α=1 and β=0



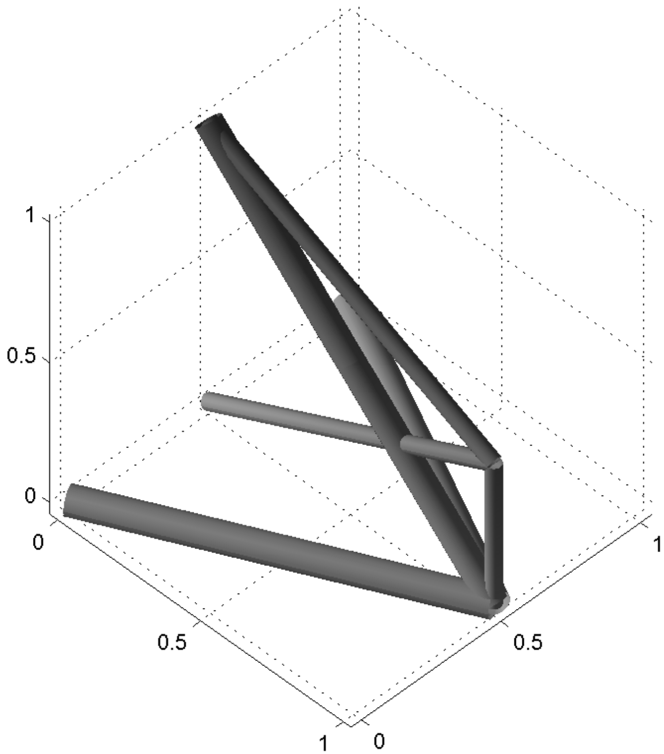
α=0 and β=1



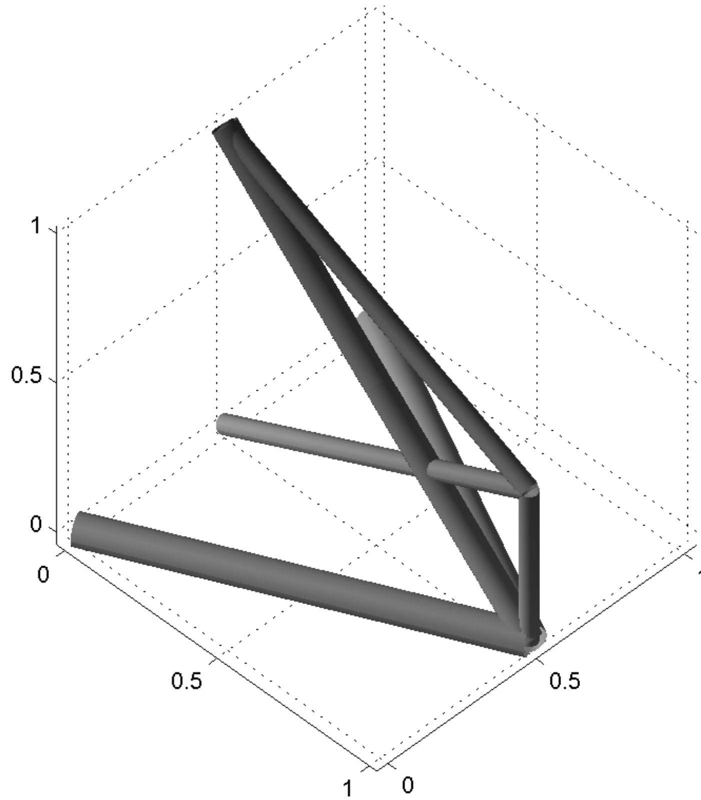

α=1 and β=0.25
1
0.5
0
0
0.5
1
0
0.5
1


α=1 and β=0.5
1
0.5
0
0
0.5
1
0
0.5
1

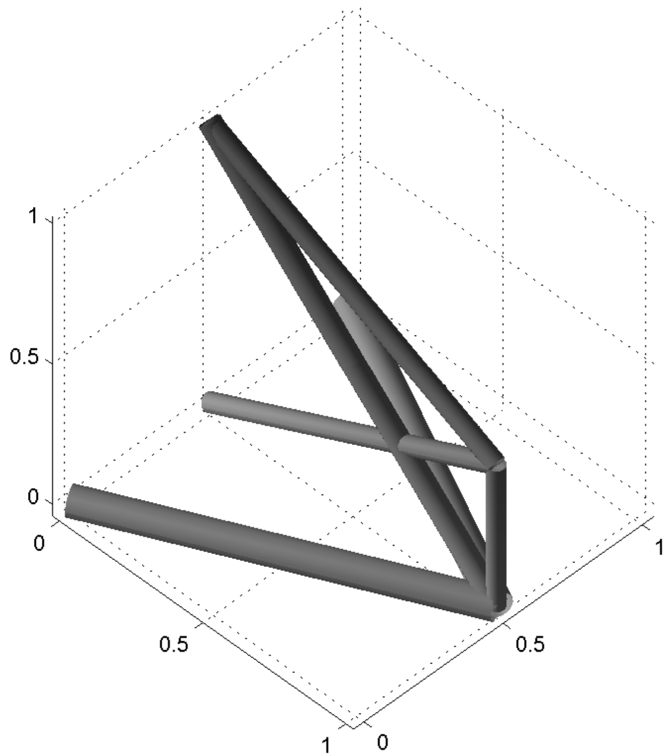
α=1 and β=0.75



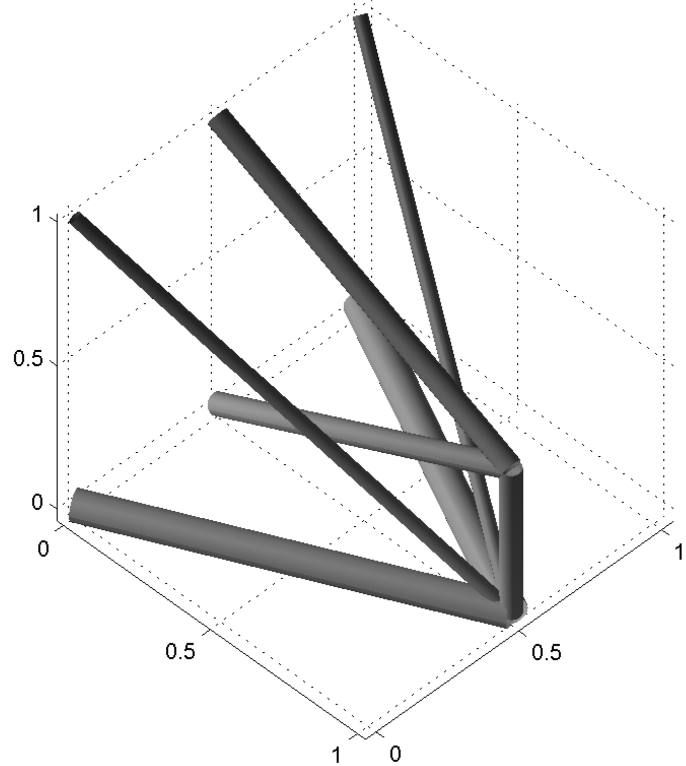
α=1 and β=1



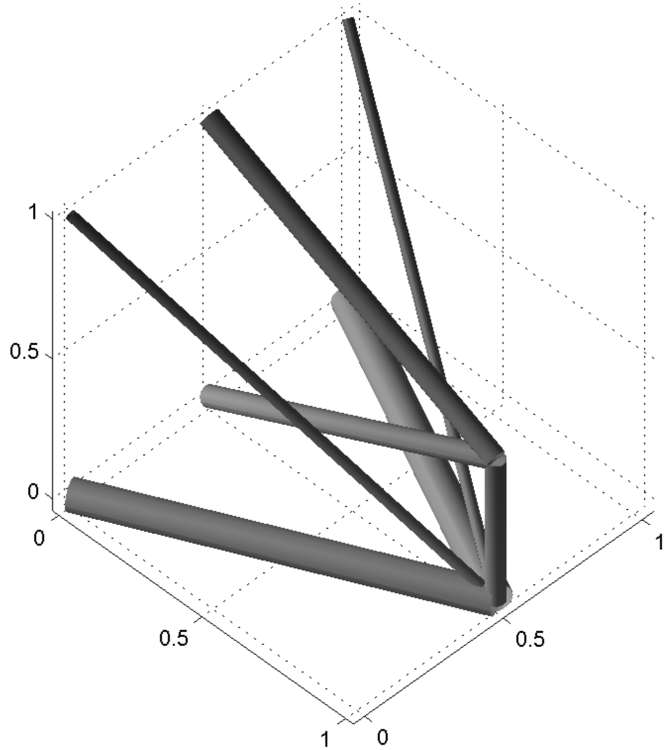
α=0.75 and β=1



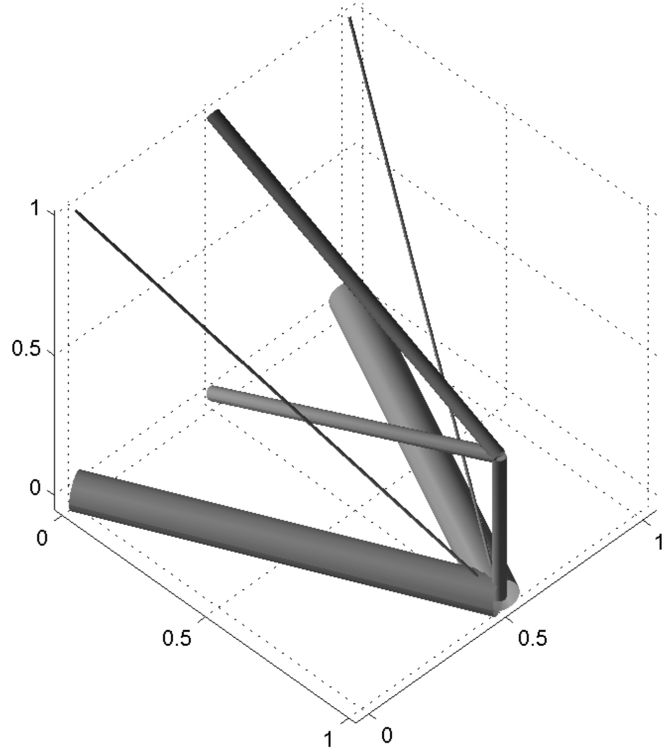
α=0.5 and β=1



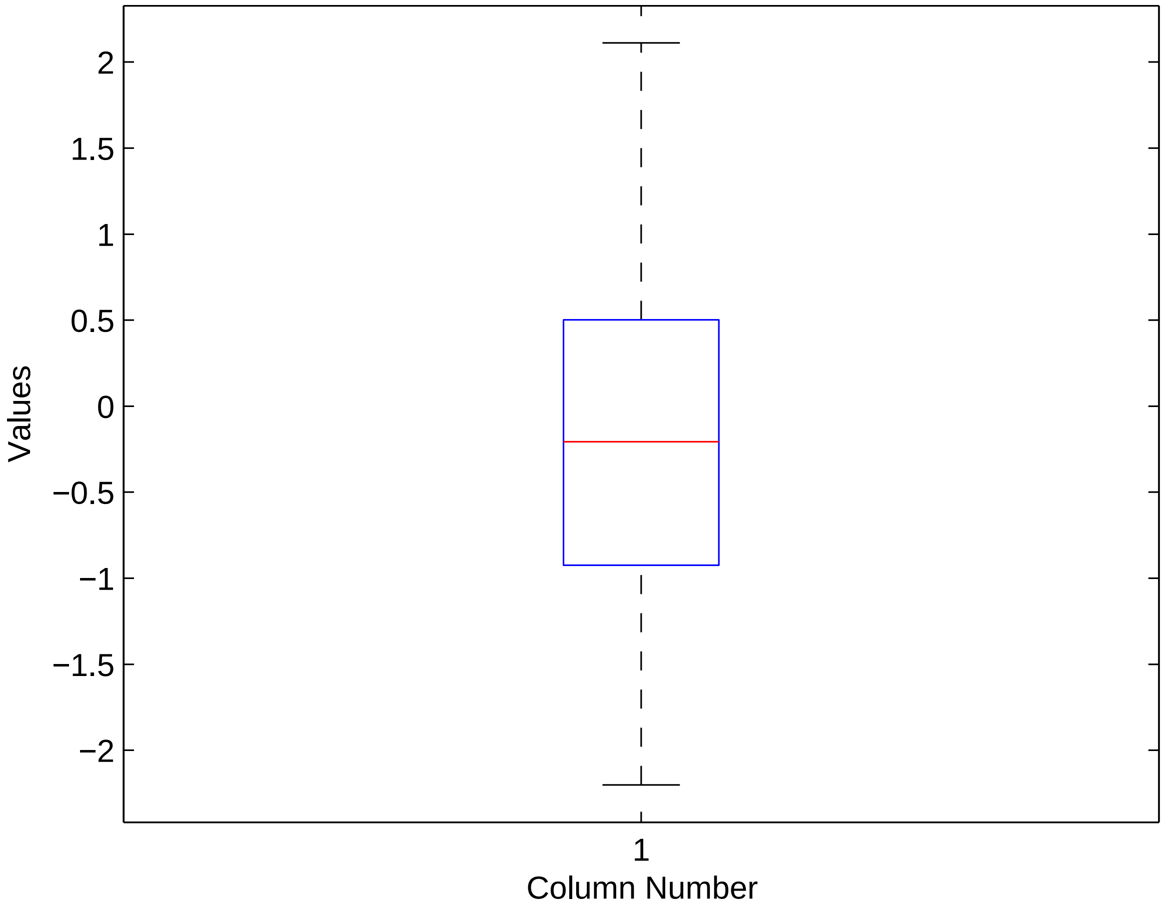
α=0.25 and β=1



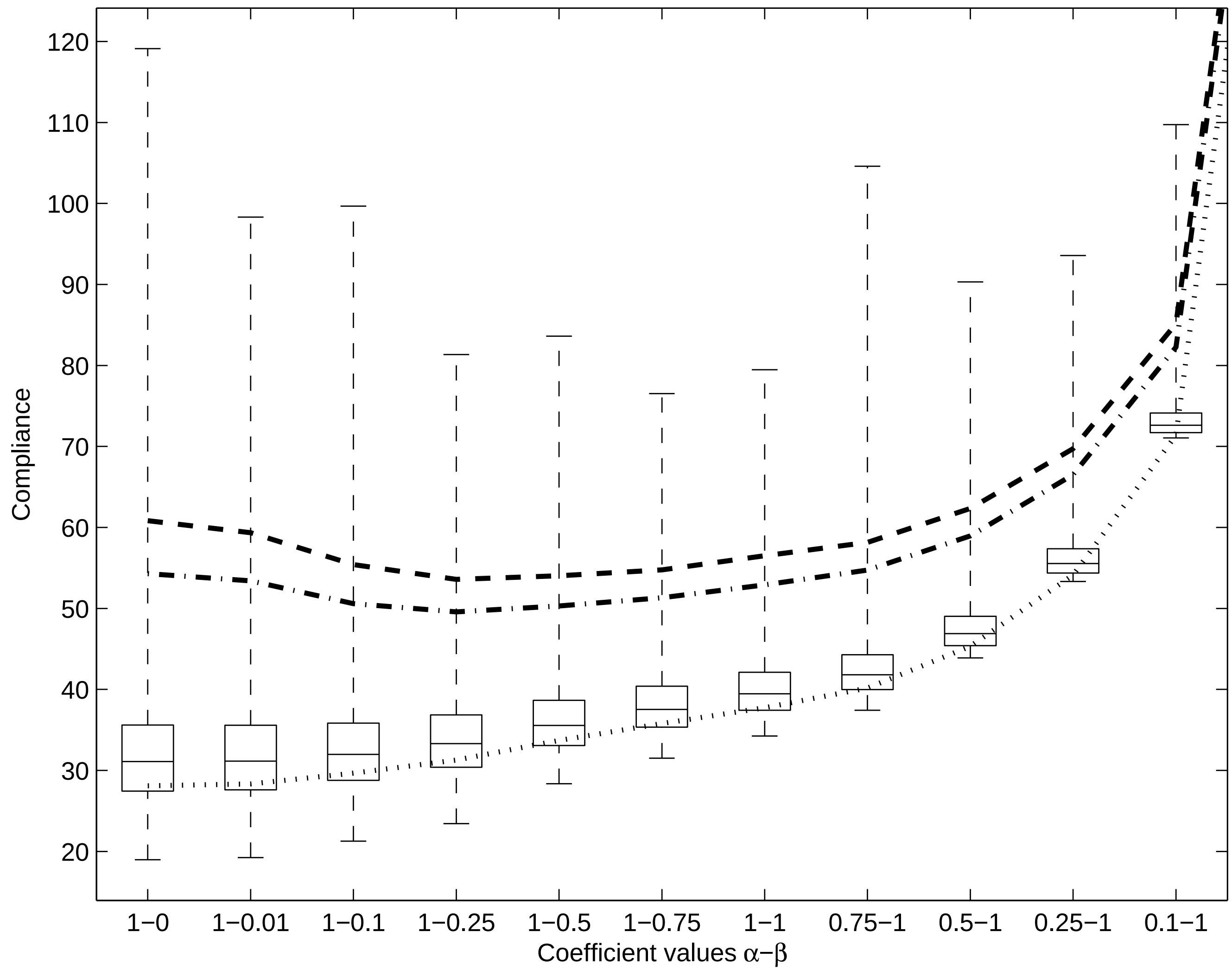
α=0.01 and β=1



Boxplot



Boxplot



120
110
100
90
80
70
60
50
40
30
20
Compliance
1−0
1−0.01
1−0.1
1−0.25
1−0.5
1−0.75
1−1
0.75−1
0.5−1
0.25−1
0.1−1
Coefficient values α−β

Conclusions

- Classical model (single load) don't get good results .
- Random model helps to avoid this problem.
- It's equivalent to multiload one, but another interpretations in weight factors holds.
- Continuous Model.
- We have to know the influence of each perturbation.
- Increase the dimension of the problem.
- Highly non linear.

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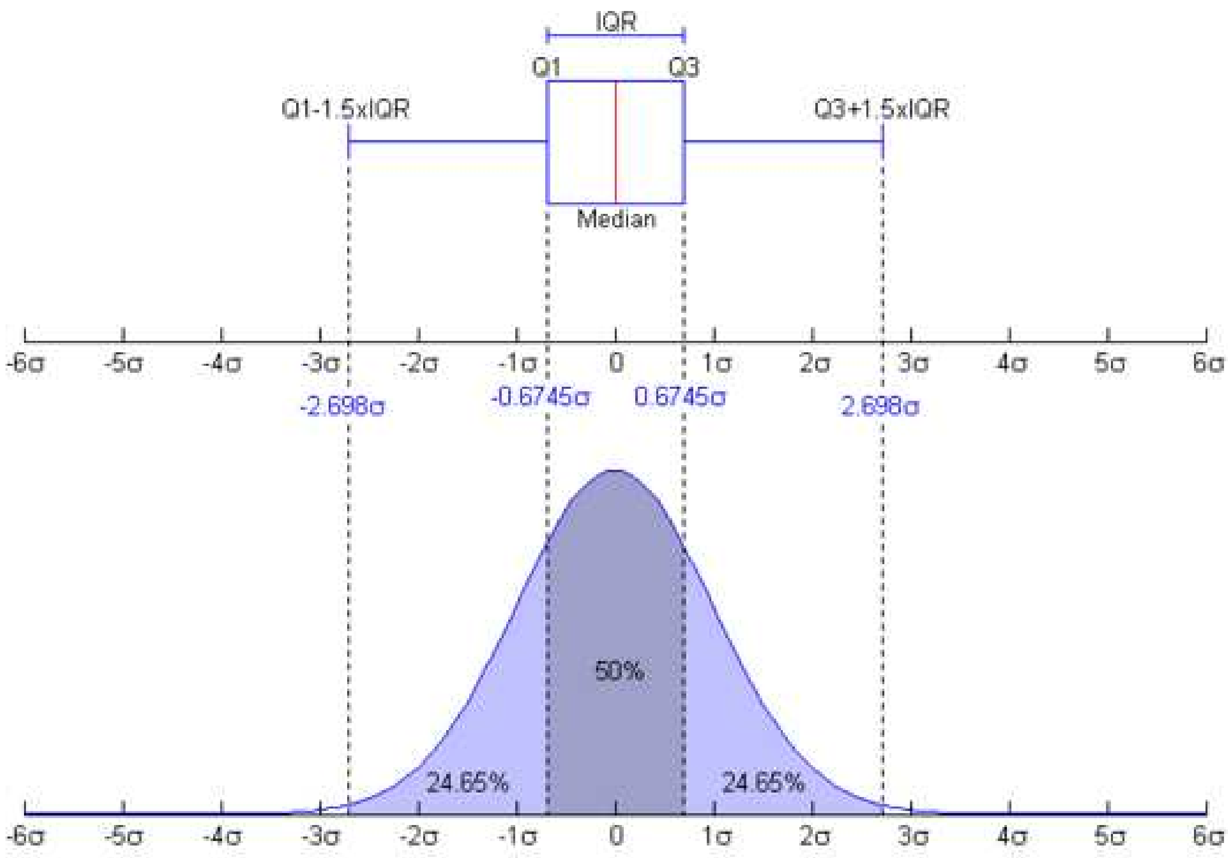

IQR
Q1
Q3
Q1-1.5xIQR
Q3+1.5xIQR
Median
-6σ -5σ -4σ -3σ -2σ -1σ 0 1σ 2σ 3σ 4σ 5σ 6σ
-2.698σ
-0.6745σ
0.6745σ
2.698σ
50%
24.65%
24.65%
-6σ -5σ -4σ -3σ -2σ -1σ 0 1σ 2σ 3σ 4σ 5σ 6σ