Selection and approximation of Cheeger sets

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Introduction

Given Ω, an open bounded subset of $\mathbb{R}^N$ and nonnegative functions f and g (more precise assumptions on the data Ω, f and g will be given later on), consider

$$\mu := h(\Omega) = \inf_{u \in BV_0} \mathcal{R}(u) \tag{1}$$

where

$$BV_0 := \{u \in BV(\mathbb{R}^N),\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}\}, \tag{2}$$

and for $u \in BV_0$ such that $\int_\Omega fu \neq 0$,

$$\mathcal{R}(u) := \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\, \mathrm{d}|Du(x)|}{\displaystyle\int_\Omega f(x)|u(x)|\, \mathrm{d}x}. \tag{3}$$

When $g = f = 1$, it is well-known that the infimum in (1) coincides with the infimum of $\mathcal{R}$ over characteristic functions of sets of finite perimeter. In this case, (1) appears as a natural relaxation of:

$$\lambda(\Omega) := \inf_{A \subset \overline{\Omega},\ \chi_A \in BV} \frac{\|D\chi_A\|(\mathbb{R}^N)}{|A|} \tag{4}$$

where $|A|$ and $\|D\chi_A\|(\mathbb{R}^N)$ denote respectively the Lebesgue measure of A and the total variation of $D\chi_A$. Problem (4) is known as Cheeger's problem, its value $\lambda(\Omega)$ is called the Cheeger constant of Ω and its minimizers are called Cheeger sets of Ω. Note also that $\lambda(\Omega)$ is the first eigenvalue of the 1-Laplacian on Ω.

Throughout the paper, we will assume that

- Ω is a nonempty open bounded subset of $\mathbb{R}^N$ with a Lipschitz boundary,
- $f \in L^\infty(\Omega)$, $f \geq f_0$ for a positive constant f_0,
- $g \in C^0(\overline{\Omega})$, $g \geq g_0$ for a positive constant g_0.

In what follows, every $u \in BV(\Omega)$ will be extended by 0 outside $\overline{\Omega}$, and thus will also be considered as an element of $BV(\mathbb{R}^N)$, still denoted u. Set, for every u in BV_0:

$$\mathcal{G}(u) := \int_{\mathbb{R}^N} g(x)\,\mathrm{d}\,|Du(x)|\,. \tag{5}$$

Since $\partial\Omega$ is Lipschitz, note that, for $u \in BV(\Omega)$:

$$\mathcal{G}(u) = \int_{\Omega} g(x)\,\mathrm{d}\,|Du(x)| + \int_{\partial\Omega} g(x)\,|u(x)|\,\mathrm{d}\,\mathcal{H}^{N-1}(x)$$

Taking advantage of the homogeneity of (1), it is convenient to reformulate (1) as the convex minimization problem

$$\mu = h(\Omega) = \inf_{u \in BV_f} \mathcal{G}(u) \tag{6}$$

where

$$BV_f := \Big\{ u \in BV(\mathbb{R}^N),\ u \geq 0,\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega},\ \int_\Omega f u = 1 \Big\}. \tag{7}$$

In analogy with the case $g = f = 1$, it is natural to consider the generalized Cheeger problem:

$$\lambda := \inf_{A \in \mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\, \mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x)\, \mathrm{d}x} = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A) \tag{8}$$

where

$$\mathcal{E} := \{A \subset \overline{\Omega} \quad \text{with} \quad \int_A f(x)\,\mathrm{d}x > 0 \quad \text{and} \quad \chi_A \in BV(\mathbb{R}^N)\}. \tag{9}$$

Again (1) can be interpreted as a relaxed formulation of (8) and one aim of the talk is to investigate the precise connections between (1) and (8). Solutions of (8) : Generalized Cheeger sets.

Some recent results:

- convergence of the first eigenvalue of the p-laplacian to the Cheeger constant as p tends to 1, convexity results (Kawohl and Fridman, 2003)
- Uniqueness of the Cheeger set when Ω is convex (Alter and Caselles, 2007 and Caselles, Chambolle, Novaga, 2006 in the smooth uniformly convex case),
- Full characterization of the Cheeger set of a convex subset of the plane (Lachand-Robert and Kawohl, 2006),
- extension to the case of a Finsler metric (Kawohl and Novaga, 2006).

Rich and well motivated topic: constrained isoperimetric problem, shape optimization, first eigenvalue and eigenfunction of the 1-laplacian, motion by mean curvature. Faber-Krahn type inequality:

$$\lambda_p(\Omega) \geq \left(\frac{\lambda(\Omega)}{p}\right)^p.$$

Recently related to landslide modelling (Ionescu and Lachand-Robert), motivation for general weights f and g that respectively represent the body forces and the (inhomogoneous) yield limit distribution of the geomaterial. In this context $h(\Omega)$ appears as a safety factor (whether $h(\Omega)$ is larger than 1 is a stability condition).

Tightly related to the continuous max flow/min cut duality theorem (Strang):

$$h(\Omega) = \max\left\{\lambda : (\lambda, v) \in \mathbb{R} \times L^{\infty}(\Omega, \mathbb{R}^d), \mathrm{div}(v) = \lambda f,\ |v| \le g\right\}$$

equivalently

$$\frac{1}{h(\Omega)} = \min\left\{\left\|\frac{v}{g}\right\|_{L^\infty} :\ v \in L^{\infty}(\Omega, \mathbb{R}^d),\ \mathrm{div}(v) = f\right\}.$$

Note the analogy with optimal transportation:

$$W_1(f_+, f_-) = \sup_{|\nabla u| \le 1} \int u(f_+ - f_-) \tag{10}$$

$$= \inf\left\{\int |v|\ : \mathrm{div}(v) = f_+ - f_-\right\}. \tag{11}$$

Existence

$$\mu = h(\Omega) = \inf_{u \in BV_f} \mathcal{G}(u) = \int_{\mathbb{R}^N} g(x) \,\mathrm{d}\, |Du(x)|\,.$$

where

$$BV_f := \left\{ u \in BV(\mathbb{R}^N),\ u \geq 0,\ u \equiv 0 \text{ on } \mathbb{R}^N \setminus \overline{\Omega}, \int_\Omega fu = 1 \right\}.$$

and Cheeger's problem:

$$\lambda := \inf_{A \in \mathcal{E}} \frac{\displaystyle\int_{\mathbb{R}^N} g(x)\,\mathrm{d}|D\chi_A(x)|}{\displaystyle\int_A f(x)\,\mathrm{d}x} = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A)$$

The direct method of the calculus of variations yields:

Proposition 1 *The infimum of (6) is achieved in* BV_f *and the infimum of (8) is achieved in* $\mathcal{E}$*.*

Outline

① Invariance

② Qualitative properties of solutions

③ Selection of Maximal Cheeger sets

④ Numerical approximation

Invariance

Proposition 2 *Let u be a solution of (6) and $H \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ be a nondecreasing function such that $H(0) = 0$. If $H \circ u \neq 0$ then $T_H(u) = H(u)/\int f H(u)$ also solves (6).*

Remark applies in particular to $H(v) = (v - t_0)_+$ and $H(v) = \min(v, t_0)$.

Remark Taking H bounded shows the existence of bounded solutions to (6). We shall see in Theorem 3 that in fact *every* solution (6) is in fact L^∞.

Theorem 1 *Let* u *be a solution of (6) and for every* $t \geq 0$, *define* $E_t := \{x \in \mathbb{R}^N \;:\; u(x) > t\}$. *For every* $t \geq 0$ *such that* E_t *has positive Lebesgue measure* $\dfrac{1}{\int_{E_t} f}\,\chi_{E_t}$ *solves (6). In particular,* $\dfrac{1}{\int_{\{u>0\}} f}\,\chi_{\{u>0\}}$ *solves (6).*

Proof:

$E_0 := \{u > 0\}$. Define for every $n \in \mathbb{N}^*$ and $v \in \mathbb{R}$:

$$H_n(v) := \begin{cases} 0 & \text{if } v \leq 0 \\ nv & \text{if } v \in [0, \frac{1}{n}] \\ 1 & \text{if } v \geq \frac{1}{n}. \end{cases}$$

$T_{H_n}(u)$ solves (6). Since $T_{H_n}(u)$ converges in $L^1(\mathbb{R}^N)$ to $\dfrac{1}{\int_{\{u>0\}} f}\,\chi_{\{u>0\}}$, we conclude that $E_0 = \{u > 0\}$ is a Cheeger set.

Let $t \geq 0$ be such that E_t has positive Lebesgue measure. From corollary 2, $v := \dfrac{(u-t)_+}{\int f(u-t)_+}$ solves (6), hence so does

$$\frac{1}{\int_{\{v>0\}} f}\,\chi_{\{v>0\}} = \frac{1}{\int_{E_t} f}\,\chi_{E_t}.$$

Converse:

Proposition 3 *Let $u \in BV_0$, $u \geq 0$. If for every $t \geq 0$ such that $E_t := \{x \in \mathbb{R}^N \ : \ u(x) > t\}$ has positive Lebesgue measure, χ_{E_t} solves (1) then u solves (1).*

Straightforward application: relaxation

Corollary 1 *The values of problems (6) and (8) coincide:*

$$\mu = \inf_{u \in BV_0} \mathcal{R}(u) = \lambda = \inf_{A \in \mathcal{E}} \mathcal{R}(\chi_A).$$

Remark One can obtain the relaxation result of corollary 1 as a direct consequence of the coarea and Cavalieri's formulae. Obviously has $\mu \leq \lambda$ and if $u \in BV_0$, $u \geq 0$, setting $E_t := \{u > t\}$, the coarea and Cavalieri's formulae yield:

$$\int_{\mathbb{R}^N} g \,\mathrm{d}|Du(x)| - \lambda \int_{\mathbb{R}^N} fu$$
$$= \int_0^\infty \left(\int_{\partial^* E_t} g \,\mathrm{d}\mathcal{H}^{N-1} - \lambda \int_{E_t} f(x)\,\mathrm{d}x \right) dt \geq 0$$

which proves that $\mu \geq \lambda$. From the previous argument, in fact, we see that the converse also holds: u solves (6) if and only if $E_t := \{u > t\}$ (which has finite perimeter for a.e. t) solves (8) for a.e. $t \geq 0$. Note that in Theorem 1, we have proved that E_t solves (8) for *all* t (and we have not used the coarea formula).

Of course, theorem 1 and its proof contain much more information than corollary 1. A more precise consequence of theorem 1 is the following

Corollary 2 $A \in \mathcal{E}$ *solves (8) if and only if there exists* u *solving (6) such that* $A = \{u > 0\}$.

A straightforward application:

Theorem 2 *Let* $(A_n)_n$ *be a sequence of solutions of (8) then* $\bigcup_n A_n$ *is also a solution of (8).*

Qualitative properties of solutions

Theorem 3 *Let u be a solution of (6). Then u belongs to $L^\infty(\Omega)$.*

Idea of the proof: Use the invariance with powers of $\min(u, M)$ and a bootstrap argument.

Combining the L^∞ estimate with the invariance property also yields:

Theorem 4 *Let u be a solution of (6), then the set $\{u = \|u\|_\infty\}$ has positive Lebesgue measure.*

Maximal Cheeger sets

In general (with weights and or in a nonconvex Ω) the Cheeger set is not unique, there are even known examples where there are infinitely many (even a continuum)! However, nonuniqueness is rather rare in the sense of Baire:

Proposition 4 *Let $g \in C^0(\overline{\Omega})$ with $g \geq g_0$ for a positive constant g_0. Then there exists a G_δ dense subset X of $C^0(\overline{\Omega}, \mathbb{R}^+)$ such that for every $f \in X$,* (6) *admits a unique solution (equivalently $\mathcal{C}$ is a singleton).*

When uniqueness fails: maximal Cheeger set.

Let us denote by $\mathcal{C}$ the set of Cheeger sets. We have seen that a (countable) union of Cheeger sets still is a Cheeger set, one then easily deduces the following:

Proposition 5 *There exists a unique maximal Cheeger set, i.e. a unique $C_0 \in \mathcal{C}$ such that for every $C \in \mathcal{C}$, C is included in C_0 up to a Lebesgue negligible set.*

Selection of maximal Cheeger sets

Natural questions:

- Do natural approximation schemes (e.g. p-laplacian) select the maximal Cheeger set at the limit?
- Do solutions of approximated problems converge to (a multiple of) the characteristic function of the maximal Cheeger set?
- Does at least their support identify the maximal Cheeger set at the limit?

Two approximation schemes:

1) p-laplacian approximation:

$$\mu_p := \sup\left\{ \int_\Omega f u \, dx \ : \ \int_\Omega g|Du|^p \, dx \le 1, \ u \in W_0^{1,p}(\Omega) \right\}. \quad (12)$$

The unique (nonnegative) maximizer u_p of (12) is of course the solution of the PDE

$$-\operatorname{div}\left(g|Du|^{p-2}Du\right) = \lambda_p f, \ u \in W_0^{1,p}(\Omega), \ \text{with } \lambda_p := \frac{1}{\mu_p}. \quad (13)$$

Does NOT select the maximal Cheeger set.

2) Concave penalization:

$$\sup\left\{ \int_\Omega f\left(u - \varepsilon\Phi(u)\right) dx \ : \ \int_{\mathbb{R}^d} g \, d|Du| \le 1, \ u \in BV_0(\Omega) \right\} \quad (14)$$

with Φ strictly convex and $\Phi(0) = 0$

Concave penalization

$$\sup\left\{\int_\Omega f\big(u-\varepsilon\Phi(u)\big)\,dx\ :\ \int_{\mathbb{R}^d} g\,d|Du|\le 1,\ u\in BV_0(\Omega)\right\}\qquad(15)$$

where $\varepsilon > 0$ is a perturbation parameter and Φ is a strictly convex nonnegative function that satisfies:

$$\Phi(0)=0,\qquad 0\le\Phi(t)<+\infty\quad\forall t\in\mathbb{R}^+.\qquad(16)$$

Theorem 5 *Let u_ε be the solution of* (15)*; then the following holds:*

- $(u_\varepsilon)_\varepsilon$ *converges in* $L^1(\Omega)$*, as* $\varepsilon \to 0^+$*, to the solution* $\overline{u}$ *of*

$$\inf\left\{\int_\Omega f\Phi(u)\,dx \;:\; u \in Q\right\}, \tag{17}$$

- $\overline{u} = \alpha\chi_{C_0}$ *for some* $\alpha > 0$ *and* $C_0 \subset \overline{\Omega}$*,*
- C_0 *is the maximal Cheeger set, i.e.* $C_0 \in \mathcal{C}$ *and* C_0 *contains every other Cheeger set (up to a Lebesgue negligible set).*

Numerical approximation

Aim : compute the maximal Cheeger set (dimension $d = 2$ or 3). Take $\Phi(t) = t^2/2$ yields the approximated problem

$$\inf\left\{\int_\Omega f\left(u - \frac{1}{\varepsilon}\right)^2 dx \;:\; G(u) \leq 1\right\}. \tag{18}$$

$$G(u) := \int_\Omega g\, d|Du| + \int_{\partial\Omega} g|u|d\mathcal{H}^{d-1}.$$

The solution of the previous problem u_ε can be expressed as

$$u_\varepsilon = \Pi_K\left(\frac{1}{\varepsilon}\right) \tag{19}$$

where Π_K denotes the projection (for the weighted L^2 inner product $(u, v) := \int_\Omega fuv$) on the closed subset K of $L^2(\Omega)$ $K := \{G \leq 1\}$.

If we further assume that $g \in C^1(\overline{\Omega})$ then it is well-known that K can be described by a set of linear constraints as follows

$$K = \left\{ u \in L^2(\Omega) \; : \; \int_\Omega \mathrm{div}(gp)u \leq 1, \; \forall p \in C^1(\Omega, \mathbb{R}^d), \; \|p\|_\infty \leq 1 \right\}. \tag{20}$$

More generally, given $u^0 \in L^2(\Omega)$, we are interested in projecting u^0 onto K i.e.

$$\inf_{u \in K} F(u) = \int_\Omega f(u - u^0)^2 \tag{21}$$

Discretization (on the square) step size $h = 1/N$, E_h the set of matrices u with entries $u_{i,j}$, $i, j \in \{0, N\}^2$, by convention we extend u by setting $u_{ij} = 0$ when either i or j belongs to $\{-1, N+1\}$. For $u = (u_{i,j})_{ij} \in E_h$ we set

$$\partial_x^h u_{i,j} := \begin{cases} h^{-1}(u_{i+1,j} - u_{i,j}) & \text{if } -1 \leq i \leq N,\ -1 \leq j \leq N-1 \\ 0 & \text{if } -1 \leq i \leq N,\ j = N. \end{cases}$$

$$\partial_y^h u_{i,j} := \begin{cases} h^{-1}(u_{i,j+1} - u_{i,j}) & \text{if } -1 \leq i \leq N-1,\ -1 \leq j \leq N \\ 0, & \text{if } -1 \leq j \leq N,\ i = N. \end{cases}$$

We also set $\nabla^h u_{i,j} = (\partial_x^h u_{i,j}, \partial_y^h u_{i,j})$. Denoting f_{ij}^h and g_{ij}^h some discrete approximation of the weights f and g (e.g. $f_{ij}^h = f(ih, jh)$, $g_{ij}^h = g(ih, jh)$) and $u_{i,j}^0$ some discretization of u^0 (approximation by mean values say) we then discretize G by defining, for all $u \in E_h$:

$$G_h(u) := h^2 \sum_{i=-1}^{N} \sum_{i=-1}^{N} g_{ij}^h |\nabla^h u_{i,j}|$$

which can be rewritten as

$$G_h(u) = h^2 \sum_{i=0}^{N-1} \sum_{i=0}^{N-1} g_{ij}^h |\nabla^h u_{i,j}|$$

$$+ h \sum_{i=0}^{N-1} \left(g_{i-1}^h |u_{i,0}| + g_{iN}^h |u_{i,N}| \right) + h \sum_{j=0}^{N} \left(g_{-1,j}^h |u_{0,j}| + g_{Nj}^h |u_{N,j}| \right).$$

Defining K_h by $K_h := \{u \in E_h \ : \ G_h(u) \leq 1\}$ and

$$F_h(u) := h^2 \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f_{ij}^h (u_{i,j} - u_{i,j}^0)^2,$$

we then approximate (21) by

$$\inf_{u \in K_h} F_h(u) \tag{22}$$

and denote by u^h the solution of (22). Denote by u^h the solution of (22). Denoting by C_{ij} the square $(ih,(i+1)h) \times (jh,(j+1)h)$, we define v_h as the piecewise constant function having value $u^h_{i,j}$ on C_{ij}. Convergence

Theorem 6 *Let v_h be defined as above, then v_h converges to $\Pi_K(u_0)$ strongly in $L^2(\Omega)$ and ∇v_h converges weakly $\star$ to $\nabla \Pi_K(u_0)$ in $\mathcal{M}(\overline{\Omega},\mathbb{R}^2)$ as $h \to 0$.*

Implementation by the Combettes and Pesquet subgradient projection iterative algorithm:

1. *Initialization.* Set $u^{(0)} = \sqrt{f} u_0$ and set $k \leftarrow 0$.
2. *Sub-gradient computation:* Define the sub-gradient of the total variation at the current iterate as

$$t^{(k)} = \frac{1}{\sqrt{f}} \operatorname{div}(g p^{(k)}) \tag{23}$$

$$\text{with} \quad p_{i,j}^{(k)} = \begin{cases} \frac{\nabla \tilde{u}_{i,j}^{(k)}}{\|\nabla \tilde{u}_{i,j}^{(k)}\|} & \text{if } \nabla \tilde{u}_{i,j}^{(k)} \neq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

where $\tilde{u} = u/\sqrt{f}$.

3. *Sub-gradient projection computation.* Define

$$z^{(k)} = \begin{cases} u^{(k)} - (G(\tilde{u}^{(k)}) - 1)\frac{t^{(k)}}{\|t^{(k)}\|^2} & \text{if} \quad G(\tilde{u}^{(k)}) > 1, \\ u^{(k)} & \text{otherwise.} \end{cases}$$

as the sub-gradient projection of the current estimate.

4. *Projection onto half-spaces.* Define the two half-spaces

$$D^{(k)} = \{v : < u^{(k)} - v, u^{(k)} - u_0 > \leq 0\},$$
$$H^{(k)} = \{v : < v - z^{(k)}, u^{(k)} - z^{(k)} > \leq 0\}.$$

The new iterate is defined as the following projection

$$u^{(k+1)} = \Pi_{D^{(k)} \cap H^{(k)}}\left(u^{(k)}\right). \tag{25}$$

5. *Boundary correction.* Constrain the current estimate to

vanish outside Ω by defining

$$u_{i,j}^{(k+1)} = \begin{cases} u_{i,j}^{(k+1)} & \text{if} \quad (i,j)/N \in \Omega, \\ 0 \text{ otherwise.} \end{cases}$$

6. *Stopping criterion.* While not converged, set $k \leftarrow k+1$ and go back to 2. If the algorithm has converged, return $u = u^{(k)}/\sqrt{f}$.