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ASYMPTOTIC BEHAVIOR OF NON LINEAR EIGENVALUE PROBLEMS INVOLVING p-LAPLACIAN TYPE OPERATORS

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Abstract. We study the asymptotic behavior of two nonlinear eigenvalue problems which involve p-Laplacian type operators. In the first problem we consider the limit as $p \to \infty$ of the sequences of the k-th eigenvalues of the $p-$Laplacian operators. The second problem we study is the homogenization of nonlinear eigenvalue problems for some p-Laplacian type operators with p fixed. Our asymptotic analysis relies on a convergence result for particular critical values of a class of Rayleigh quotients, stated in a unified framework, and on the notion of Γ-convergence.

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1. Introduction

An eigenvalue of the p-Laplacian is a real number $\lambda \in \mathbb{R}$ such that the problem

$$\begin{cases} -div(|Du|^{p-2}Du) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one non trivial solution in $W_0^{1,p}(\Omega)$. Here solution is intended in the distributional sense and Ω is assumed to be a regular, bounded, open subset of $\mathbb{R}^N$. One easily proves that λ is an eigenvalue if and only if it is a critical value of the Rayleigh quotient $\frac{\|\nabla v\|^p_{L^p}}{\|v\|^p_{L^p}}$. The k-th eigenvalue is obtained by the classical Ljusternik-Schnirelman theory (see [13, 18, 19] for a detailed description), and it is defined as follows

$$(\lambda_p^k)^p \ := \ \inf \left\{ \sup_{v \in G} \frac{\|\nabla v\|^p_{L^p}}{\|v\|^p_{L^p}} \ : \ G \subset L^p, \ genus(G) \geq k \right\}$$

where $genus(G)$ is the Krasnoselskii genus of G (for a precise statement and presentation we refer to section §2).

In this paper, we study the asymptotic behavior of the sequence of the k-th eigenvalues associated with families of monotone operators of $p-$Laplacian type. The study of this type of problem arises in different settings for various applications, see for example [5], [9], [25] and the references therein. A natural way to deal with this asymptotic problem in the linear case is the study of the convergence of the resolvent operator (see [16] ch. 10, lemma XI 9.5, more reference on the linear case can be found in [5]). The present work is motivated by the study of two asymptotic behavior problems involving sequences of $p-$laplacian type operators for which we propose a unified approach in the general setting of the convergence of particular critical values of a class of Rayleigh quotients.

The first problem we consider is the study of the asymptotic behavior as $p \to \infty$ of the k-th non-linear eigenvalue of the p-Laplacian operator. This problem was in particular studied in [21] and [20], where the convergence of the first two eigenvalues was examined in details and partial results and conjectures for higher order eigenvalues were given. Our main contribution to this problem is the proof of the convergence for the generalized sequence of the k-th eigenvalues (suitably renormalized) for any positive integer k and a variational characterization of this limit (see §4, Theorem 4.3).

The second problem we consider is the asymptotic behavior of k-th eigenvalues associated with a family $(\mathcal{A}_\varepsilon)_\varepsilon$ of p-Laplacian type operator $\mathcal{A}_\varepsilon(v) = -div(a_\varepsilon(\cdot, \nabla v(\cdot)),$ with fixed p. Assuming that $\mathcal{A}_\varepsilon$ derives from a convex and p-homogeneous integral functional $F_\varepsilon(v) = \int_\Omega f_\varepsilon(x, \nabla v(x))dx$, the k-th eigenvalue λ_ε^k of $\mathcal{A}_\varepsilon$ is given by

$$\lambda_\varepsilon^k \ := \ \inf \left\{ \sup_{v \in G} \frac{F_\varepsilon(v)}{\|v\|_{L^p}^p} \ : \ G \subset L^p, \ genus(G) \geq k \right\},$$

so that λ_ε^k is a particular critical point of the Rayleigh quotient $\frac{F_\varepsilon(v)}{\|v\|_{L^p}}$ (we also refer to section §2 for a more precise presentation). The main contribution we give to the problem of this asymptotic study is a positive answer to a question raised in [5]. In this last work it was proved that the limit of any sequence of eigenvalues is an eigenvalue of the limit problem and that the sequence of the first eigenvalues (i.e. when $k = 1$) converges to the first eigenvalue of the limit operator. It follows from the present work that the sequence of the k-th eigenvalues converges to the k-th eigenvalue for any greater k (see §5, Theorem 5.1).

These two problems of asymptotic behavior have the same structure

$$\lambda_n \ := \ \inf \left\{ \sup_{v \in G} \frac{F_n(v)}{\|v\|_{L^{p_n}}} \ : \ G \subset L^{p_n}, \ genus(G) \geq k \right\},$$

where the family $(F_n)_n$ converges to some limit functional. The technique we use to study these problems is based on the notion of Γ-convergence (see the next section for details). The Γ-convergence was introduced in [14] to deal with convergence of minimizers of sequences of functionals. Here we deal with some critical values of sequences $(F_n)_n$ of functionals which $\Gamma-$converge. This is done by introducing a new sequence of functionals whose arguments are compact sets of the L^p spaces, and by endowing with the Hausdorff metric the set of compact subsets of the suitable space. This gives rise to a unified treatment in a general framework of the two problems mentioned above (see §3, Theorem 3.3).

A relative advantage in dealing with these problems using the Γ-convergence is that we do not need to have a limit operator and that we know that the limit critical value is of saddle type. Let us finally underline that our method gives a general scheme to deal with Γ-convergence and critical points obtained via an index theory.

2. Definitions and preliminary results

2.1. Γ-convergence.

Let X be a metric space, a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge to F_∞ at x if

$$F_\infty(x) \ = \ \Gamma - \liminf F_n(x) \ = \ \Gamma - \limsup F_n(x), \tag{2.1}$$

where

$$\begin{cases} \Gamma - \liminf F_n(x) \ = \ \inf \left\{ \liminf F_n(x_n) : \ \ x_n \to x \ in \ X \right\}, \\ \Gamma - \limsup F_n(x) \ = \ \inf \left\{ \limsup F_n(x_n) : \ \ x_n \to x \ in \ X \right\}. \end{cases} \tag{2.2}$$

The $\Gamma-$convergence was introduced in [14], for an introduction to this theory we refer to [15] and [4]. The following is a variation of a classical theorem which reports properties of Γ-convergence that we shall use in the following.

Theorem 2.1. *Assume that the sequence $(F_n)_{n\in\mathbb{N}}$ of functionals is such that*

$$\Gamma - \liminf_{n\to+\infty} F_n \;\geq\; F_\infty \qquad \text{on } X,$$

where F_∞ is lower-semicontinuous on X. Assume in addition that the sequence $(F_n)_n$ is equi-coercive on X, then

(1) $\liminf_{n\to+\infty}\left(\inf_{x\in X} F_n(x)\right) \geq \inf_{x\in X} F_\infty(x)$,

(2) *if* $\lim_{n\to+\infty}\left(\inf_{x\in X} F_n(x)\right) = \inf_{x\in X} F_\infty(x)$, *then one has* $F_\infty(x_\infty) = \inf_{x\in X} F_\infty(x)$ *for any cluster point x_∞ of a sequence $(x_n)_{n\in\mathbb{N}}$ such that*

$$\forall n\in\mathbb{N} \qquad F_n(x_n) \leq \inf_{x\in X} F_n(x) + \varepsilon_n$$

with $\varepsilon_n \to 0$ as $n\to\infty$.

2.2. Krasnoselskii genus.

We recall here some basic facts about the Krasnoselskii genus which will be used in what follows. We refer to [26, 27] for a more complete introduction on the index theories.

Definition 2.2. *Let E be a real Banach space and let $A\subset E$ be a nonempty closed symmetric set (i.e. $A=-A$). The genus $\gamma(A)$ of the set A is the integer defined as*

$$\inf\{m\in\mathbb{N} \;:\; \text{there exists a continuous and odd mapping } \varphi : A\to\mathbb{R}^m\setminus\{0\}\},$$

where the above infimum is assumed to be $+\infty$ if the set above is empty.

Remark 2.3. The following elementary properties hold:

(1) If 0 belongs to A then $\gamma(A)=+\infty$,
(2) $A_1\subset A_2$ implies $\gamma(A_1)\leq\gamma(A_2)$.

We will need the following continuity property of the genus (see Proposition 5.4 of [27] or Proposition 7.5 of [26] for a proof).

Proposition 2.4. *Assume that $A\subset E$ is compact, then there is a symmetric neighborhood N of A in E such that $\gamma(\overline{N})=\gamma(A)$.*

2.3. Hausdorff convergence of compact sets.

Let (X,d) be a metric space, and denote by $\mathcal{K}$ the set of the compact subsets of X. The distance $d_{\mathcal{H}}:\mathcal{K}\times\mathcal{K}\to\mathbb{R}_+$ is defined as

$$d_{\mathcal{H}}(G,F) := \sup_{x\in G} d(x,F) + \sup_{y\in F} d(y,G).$$

It is easy to check that $(\mathcal{K},d_{\mathcal{H}})$ is a metric space and to check the following property (a reference for this part is [1])

Proposition 2.5. *If X is compact then $(\mathcal{K},d_{\mathcal{H}})$ is compact.*

As an application of Proposition 2.5 we obtain two examples which will be useful in the sequel of the paper.

Example 2.6. Let Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary. By a standard application of the Sobolev compact embedding theorems we get:

- Let $p \in [1, \infty)$ and $q \geq 1$ such that $q \in [N, \infty]$ or $q^* := \frac{Nq}{N-q} > p$, then the set
$$\{G \subset W_0^{1,q}(\Omega) \mid G \text{ is closed and bounded by } C \text{ in } W_0^{1,q}(\Omega)\}$$
equipped with the Hausdorff distance induced by the L^p norm is compact for any constant $C > 0$.
- Let $q \in (N, \infty]$ and $C > 0$, then the set
$$\{G \subset W_0^{1,q}(\Omega) \mid G \text{ is closed and bounded by } C \text{ in } W_0^{1,q}(\Omega)\}$$
equipped with the Hausdorff distance induced by the sup norm of $\mathcal{C}_0$ is compact.

We will need the following characterization of the Hausdorff convergence which can be obtained directly from the definition:

Proposition 2.7. *If $(K_n)_n$ is a sequence in $\mathcal{K}$, then $K_n \to K$ with respect to $d_{\mathcal{H}}$ if and only if*

(1) *for each sequence $(x_n)_n$ such that $x_n \in K_n$ for all n, any accumulation point $x \in X$ for $(x_n)_n$ belongs to K,*
(2) *for each point $x \in K$ we can find a sequence x_n with $x_n \in K_n$ converging to x.*

The next lemma of elementary proof will be useful to study the behavior of the genus with respect to the Hausdorff convergence of compact sets.

Lemma 2.8. *Let $(K_n)_n$ is a sequence in $\mathcal{K}$ converging to K with respect to $d_{\mathcal{H}}$. Then any open set $A \subset X$ which contains K also contains K_n for n large enough.*

2.4. **Nonlinear eigenvalues of p-Laplacian type operators.**
In this section we introduce the basic definition for the eigenvalues of the p-Laplacian and some generalization to $p-$Laplacian type operators.

From now on, Ω denotes a bounded connected open subset of $\mathbb{R}^N$ with Lipschitz boundary. In the following, p is a real number in $]1, +\infty[$ and we shall denote by $\|.\|_p$ the usual norm of $L^p(\Omega)$ (or $L^p(\Omega; \mathbb{R}^N)$ when dealing with the gradient of some element of $W_0^{1,p}(\Omega)$).

An eigenvalue of the $p-$Laplacian operator $-\Delta_p$ is a real number λ for which the problem

$$\begin{cases} -\Delta_p u := -div(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a non-zero solution in $W_0^{1,p}(\Omega)$. This problem (and its generalizations to monotone elliptic operators) has been widely studied in the literature and for more detailed treatment we refer to [2, 8, 13, 18, 19, 20, 24]. Much is still unknown about the eigenvalues of the $p-$Laplacian operator. However let us report some of the known results which will be relevant for this paper. Every eigenvalue is a critical value for the Rayleigh quotient

$$v \mapsto \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx} \quad \left(= \frac{\|\nabla v\|_p^p}{\|v\|_p^p} \right)$$

which is a Gateaux differentiable functional on $W_0^{1,p}(\Omega)$ outside the origin. Moreover, a sequence $(\lambda_p^k)_{k\geq 1}$ of eigenvalues can be obtained as follows (we refer to [18] and [24] for details). Denote by $\Sigma_p^k(\Omega)$ the set of those subsets G of $W_0^{1,p}(\Omega)$ which are symmetric (i.e. $G = -G$), contained in the set $\{v : \|v\|_p = 1\}$, strongly compact in $W_0^{1,p}(\Omega)$ and such that $\gamma(G) \geq k$, and set

$$\lambda_p^k = \inf_{G\in\Sigma_k^p(\Omega)} \sup_{u\in G} \|\nabla u\|_p^p.$$

Then each λ_p^k defined as above is an eigenvalue of the p-Laplacian operator and $\lambda_p^k \to +\infty$ as $k \to \infty$. Moreover it is known that λ_p^1 is the smallest eigenvalue of $-\Delta_p$, that it is simple (see [7] for a short proof) and that the operator $-\Delta_p$ doesn't have any eigenvalue between λ_p^1 and λ_p^2.

As a consequence of our results (see Corollary 3.6), we shall get that the above definition of the k-th eigenvalue may be rewritten

$$\lambda_p^k = \min_{G\in\mathcal{G}_k^p(\Omega)} \sup_{u\in G} \|\nabla u\|_p^p. \tag{2.3}$$

In the above formula $\mathcal{G}_p^k(\Omega)$ is the set of those subsets G of $W_0^{1,p}(\Omega)$ which are symmetric, contained in the set $\{v : \|v\|_p = 1\}$, closed and bounded in $W_0^{1,p}(\Omega)$ (and thus compact in $L^p(\Omega)$) and such that $\gamma(G) \geq k$, where $\gamma(G)$ is the genus of G as a subset of $L^p(\Omega)$.

More generally, consider a p-Laplacian type operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p}(\Omega)$ of the form $\mathcal{A}(u) = -div(a(x,\nabla u(x)))$ where the function $a : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies

- **(h1)** a is a Carathéodory function i.e. $a(x,\cdot)$ is a continuous function for a.e. $x \in \Omega$ and $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^n$,
- **(h2)** $a(x,\cdot)$ is positively homogeneous of degree $p-1$ for a.e. x,
- **(h3)** $a(x,\cdot)$ is odd for a.e. x
- **(h4)** a is cyclically monotone, i.e.

$$\sum_{i=1}^{m} \langle a(x,\xi_i), \xi_{i+1} - \xi_i\rangle \leq 0$$

for a.e. $x \in \Omega$, any $m \geq 2$ and $\xi_1,\dots,\xi_m \in \mathbb{R}^N$ (with $\xi_{m+1} = \xi_1$)
- **(h5)** there exists $\beta > \alpha > 0$ such that the following growth conditions hold

$$\alpha|\xi|^p \leq \langle a(x,\xi),\xi\rangle \qquad \textit{and} \qquad |a(x,\xi)| \leq \beta|\xi|^{p-1}$$

for all $\xi \in \mathbb{R}^N$ and for a.e. $x \in \Omega$.

Then an eigenvalue λ for $\mathcal{A}$ is a real number for which the problem

$$\begin{cases} -div(a(x,\nabla u(x))) = \lambda|u(x)|^{p-2}u(x) & \text{for a.e. } x \text{ in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-trivial distributional solution in $W_0^{1,p}(\Omega)$. Under the hypothesis (h1-5), there exists an integrand $f : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ satisfying the following assumptions (we refer to Lemma 3.1 and Proposition 3.2 of [5]):

- **(a1)** f is a Carathéodory function,
- **(a2)** $f(x,\cdot)$ is convex, differentiable with gradient $a(x,\cdot)$,
- **(a3)** $f(x,\cdot)$ is positively homogeneous of degree p,
- **(a4)** $f(x,\cdot)$ is even,

(a5) the following growth conditions holds

$$\alpha|\xi|^p \leq f(x,\xi) \leq \beta|\xi|^p$$

for all $\xi \in \mathbb{R}^N$ and for a.e. $x \in \Omega$,

Then for any integer $k \geq 1$ one can define the k-th eigenvalue of $\mathcal{A}$ as being

$$\lambda^k := \inf_{G \in \mathcal{G}_k^p(\Omega)} \sup_{u \in G} \int_\Omega f(x, \nabla u)dx. \tag{2.4}$$

Then, part of the study of the homogenization process for the eigenvalue problems associated to a family $\mathcal{A}_\varepsilon := -div(a_\varepsilon(\cdot,\cdot))$ of monotone elliptic operator of $p-$Laplacian type reduces to the study of the limit of a family of problems like (2.4).

3. A general convergence result

In this section we state and prove the main result of the paper. In the following, $(F_\varepsilon)_{\varepsilon \geq 0}$ is a family of functionals defined on $L^1(\Omega)$ with values in $[0,+\infty]$ such that:

(A1) for any $\varepsilon > 0$, F_ε is convex and 1-homogeneous;

(A2) there exists $\beta > \alpha > 0$ such that for any $\varepsilon > 0$ there exists $p_\varepsilon \in [1,+\infty]$ for which

$$\begin{cases} \alpha\|\nabla v\|_{p_\varepsilon} \leq F_\varepsilon(v) \leq \beta\|\nabla v\|_{p_\varepsilon} & \text{if } v \in W_0^{1,p_\varepsilon}(\Omega), \\ F_\varepsilon(v) = +\infty & \text{otherwise;} \end{cases}$$

(A3) the family $(p_\varepsilon)_{\varepsilon>0}$ converges to some $p_0 \in [1,+\infty]$ and the family $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges in $L^{p_0}(\Omega)$ (or $C_0(\Omega)$ if $p_0 = +\infty$) to some functional F_0.

Notice that the functional F_0 also satisfies (A1) and (A2).

Remark 3.1. In the case $p_\varepsilon = +\infty$, the notation $W_0^{1,\infty}(\Omega)$ stands for $W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$. Notice that this last space strictly contains the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,\infty}(\Omega)$, and is thus larger than the space usually denoted by $W_0^{1,\infty}(\Omega)$: the reason for our notation is that $W^{1,\infty}(\Omega) \cap \mathcal{C}_0(\Omega)$ is the natural limit space for $W_0^{1,p}(\Omega)$ as $p \to +\infty$. In the following, the notation $\mathcal{G}_\infty^k(\Omega)$ generalizes that given in §2.4.

We now define a common framework for the study of the non-linear eigenvalues of the family $(F_\varepsilon)_\varepsilon$. In the rest of this section, we shall denote by $\mathcal{K}_s(\Omega)$ the set of compact symmetric subsets of $L^{p_0}(\Omega)$ (or $\mathcal{C}_0(\Omega)$ if $p_0 = +\infty$), and by $d_\mathcal{H}$ the Hausdorff distance induced on $\mathcal{K}_s(\Omega)$ by the usual norm of $L^{p_0}(\Omega)$.

Lemma 3.2. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$, the set $\mathcal{G}_{p_\varepsilon}^k(\Omega)$ is included in $\mathcal{K}_s(\Omega)$. Moreover, the genus of an element $G \in \mathcal{G}_{p_\varepsilon}^k(\Omega)$ is the same as its genus as an element of $\mathcal{K}_s(\Omega)$.*

Proof. We first consider the case $p_0 \in [1,+\infty[$: as $p_\varepsilon \to p_0$, there exists $\varepsilon_0 > 0$ such that $p_\varepsilon > \frac{N}{N+p_0}p_0$ for any $\varepsilon < \varepsilon_0$. For such ε, since an element G of $\mathcal{G}_{p_\varepsilon}^k(\Omega)$ is closed and bounded in $W_0^{1,p_\varepsilon}(\Omega)$ we infer from the Sobolev compact embedding theorems that G is in fact a compact subset of $L^{p_0}(\Omega)$. In the case $p_0 = +\infty$, it is sufficient to take $\varepsilon_0 > 0$ such that $p_\varepsilon \geq N+1$ for any $\varepsilon < \varepsilon_0$.

Let $\varepsilon < \varepsilon_0$, and assume that $p_\varepsilon \geq p_0$: then the identity mapping $i : L^{p_\varepsilon}(\Omega) \to L^{p_0}(\Omega)$ is continuous, and since G is compact in $L^{p_\varepsilon}(\Omega)$, the sets G and $i(G)$ are homeomorphic so that the genus of G as as subset of $L^{p_\varepsilon}(\Omega)$ is the same as its genus as a subset of $L^{p_0}(\Omega)$. When $p_\varepsilon \le p_0$, the same argument works with the identity mapping $i : L^{p_0}(\Omega) \to L^{p_\varepsilon}(\Omega)$. □

For any integer $k \ge 1$ and number $\varepsilon \ge 0$, we associate to F_ε the functional $J^k_\varepsilon : \mathcal{K}_s(\Omega) \to [0, +\infty]$ given by

$$J^k_\varepsilon(G) := \begin{cases} \sup\limits_{v \in G} F_\varepsilon(v) & \text{if } G \in \mathcal{G}^k_{p_\varepsilon}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Generalizing the definition (2.3) introduced in §2.4, we define the k-th eigenvalue of the functional F_ε as

$$\lambda^k_\varepsilon := \inf_{G \in \mathcal{G}^k_{p_\varepsilon}} \sup_{v \in G} F_\varepsilon(v)$$

and we deduce from Lemma 3.2 that for ε small enough this can be rewritten

$$\lambda^k_\varepsilon = \inf \left\{ J^k_\varepsilon(G) : G \in \mathcal{K}_s(\Omega) \right\}.$$

We can now state the main result of this section, which in particular yields that $\lambda^k_\varepsilon \to \lambda^k_0$ as $\varepsilon \to 0$:

Theorem 3.3. *Let k be a positive integer, and assume that the family $(F_\varepsilon)_{\varepsilon>0}$ satisfies (A1-3). Then there exists $\varepsilon_0 > 0$ such that the family $(J^k_\varepsilon)_{0 \le \varepsilon < \varepsilon_0}$ is equicoercive on $(\mathcal{K}_s(\Omega), d_{\mathcal{H}})$,*

$$\Gamma - \liminf_{\varepsilon \to 0} J^k_\varepsilon \ \ge\ J^k_0 \quad \text{on } \mathcal{K}_s(\Omega)$$

and

$$\lim_{\varepsilon \to 0} \left(\inf_{G \in \mathcal{K}_s(\Omega)} J^k_\varepsilon(G) \right) = \inf_{G \in \mathcal{K}_s(\Omega)} J^k_0(G).$$

Proof. We divide the proof in three steps.
Step 1. Equicoercivity. We first consider the case $p_0 \in [1, +\infty)$, and define an exponent $q \ge 1$ depending on the dimension N and p_0 as follows:

$$q(P_0, N) := q := \begin{cases} 1 & \text{if } N = 1 \text{ or } N > 1 \text{ and } p_0 \in [1, \frac{N}{N-1}], \\ \frac{Np_0}{N+p_0} & \text{if } p_0 \in (\frac{N}{N-1}, +\infty). \end{cases}$$

Let $\delta = \frac{p_0 - q}{2}$ and $\varepsilon_0 > 0$ such that for any $\varepsilon < \varepsilon_0$ we have $p_\varepsilon \ge q + \delta$ (notice that for $p_0 = 1$ one has $\delta = 0$ thus $p_\varepsilon \ge q$ for any $\varepsilon > 0$). Observe that the critical exponent $(q+\delta)^*$ is always strictly greater than p_0.

Let $\varepsilon < \varepsilon_0$ and $G_\varepsilon \in \mathcal{K}_s(\Omega)$ be such that $J^k_\varepsilon(G_\varepsilon) \le C$. By definition of J^k_ε and property (A2) the estimate $\|u\|_{W^{1,p_\varepsilon}_0(\Omega)} \le \frac{C}{\alpha}$ holds for all $u \in G_\varepsilon$ and this implies

$$\|u\|_{W^{1,q+\delta}_0(\Omega)} \le |\Omega|^{\frac{1}{q+\delta} - \frac{1}{p_\varepsilon}} \frac{C}{\alpha} \le K$$

where K is a constant independent of $\varepsilon < \varepsilon_0$. By Proposition 2.5 and the first part of Example 2.6, the sublevels $\{J_\varepsilon \le C\}$ are contained in a common compact subset of $(\mathcal{K}_s(\Omega), d_{\mathcal{H}})$ for $\varepsilon < \varepsilon_0$, so that the family $(J^k_\varepsilon)_{0 \le \varepsilon < \varepsilon_0}$ is equicoercive.

In the case $p_0 = \infty$ we follow the same scheme but we choose ε_0 such that $p_\varepsilon \ge N + 1$ for $\varepsilon < \varepsilon_0$ and use the second part of Example 2.6.

Step 2. We show the Γ-liminf estimate. To this end, let $G \in \mathcal{K}_s(\Omega)$ and $(G_\varepsilon)_{\varepsilon>0}$ be a family such that $G_\varepsilon \overset{d_{\mathcal{H}}}{\to} G$, we shall prove that

$$\liminf_{n \to \infty} J^k_\varepsilon(G_\varepsilon) \ \ge\ J^k_0(G).$$

Without loss of generality, we may assume that there exists a constant $C > 0$ such that $J^k_\varepsilon(G_\varepsilon) \leq C$ for any $\varepsilon > 0$. Let us first show that $\gamma(G) \geq k$. By Proposition 2.4 there exists an open symmetric neighborhood N of G in $L^{p_0}(\Omega)$ (or $\mathcal{C}_0(\Omega)$ for $p_0 = +\infty$) such that $\gamma(\bar{N}) = \gamma(G)$. We then infer from Lemma 2.8 that $G_\varepsilon \subset N \subset \bar{N}$ for ε small enough. By the second property in Remark 2.3, for such an $\varepsilon > 0$ we get

$$k \leq \gamma(G_\varepsilon) \leq \gamma(\bar{N}) = \gamma(G).$$

Let now $u \in G$, by the sequential characterization of the Hausdorff convergence of compact sets (Proposition 2.7), there exists a (generalized) sequence $u_\varepsilon \in G_\varepsilon$ which converges to u in $L^{p_0}(\Omega)$. By the $\Gamma-$liminf inequality for the functionals F_ε (assumption (A3))

$$F_0(u) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon\to 0}\left(\sup_{G_\varepsilon} F_\varepsilon\right) = \liminf_{\varepsilon\to 0} J^k_\varepsilon(G_\varepsilon).$$

Taking the supremum for $u \in G$ gives the claim.

Step 3. By the two previous steps and Theorem 2.1, we infer that it only remains to prove that

$$\limsup_{\varepsilon\to 0}\left(\inf_{G\in\mathcal{K}_s(\Omega)} J^k_\varepsilon(G)\right) \leq \inf_{G\in\mathcal{K}_s(\Omega)} J^k_0(G).$$

We assume that $\inf_{G\in\mathcal{K}_s(\Omega)} J^k_0(G) < +\infty$, otherwise there is nothing to prove.

We fix $\delta \in]0,1[$ and first study the case $p_0 \in]1,+\infty]$. Let $G_0 \in \mathcal{K}_s(\Omega)$ be such that

$$\inf_{G\in\mathcal{K}_s(\Omega)} J^k_0(G) \;\;\geq\;\; J^k_0(G_0) - \delta.$$

Since G_0 is compact in $L^{p_0}(\Omega)$, there exists a finite family $(u^i)_{1\leq i\leq m}$ in G_0 such that

$$G_0 \subset \bigcup_{i=1}^m B_{L^{p_0}(\Omega)}(u^i, \frac{\delta}{5})$$

We infer from hypothesis (A3) that for any $i \in \{1,\dots,m\}$ there exists a family $(u^i_\varepsilon)_\varepsilon$ in $L^{p_0}(\Omega)$ such that

$$u^i_\varepsilon \to u^i \text{ in } L^{p_0}(\Omega) \;\text{ and }\; F_\varepsilon(u^i_\varepsilon) \to F_0(u^i) \;\text{ as }\; \varepsilon \to 0.$$

Taking ε_0 as in step 1, for any $\varepsilon \in]0,\varepsilon_0[$ we define C_ε to be the convex closure of the finite symmetric set $\{\pm u^i_\varepsilon : i = 1,\dots,m\}$. We may assume that $F_\varepsilon(u^i_\varepsilon) < +\infty$ for any i and any such ε, so that the finite dimensional set C_ε is a compact convex subset of $W^{1,p_\varepsilon}_0(\Omega)$ and $L^{p_0}(\Omega)$. Now let $1 < q < p_0$ be such that

$$\forall i \in \{1,\dots,m\} \qquad \|u_i\|_q \geq 1 - \frac{\delta}{5}.$$

We denote by P_ε the projection onto C_ε for the norm of $L^q(\Omega)$, for which C_ε is also compact. Then we notice that for any $v \in G_0$ there exists $i \in \{1,\dots,m\}$ such that $\|v - u^i\|_{p_0} \leq \frac{\delta}{5}$, therefore

$$\begin{aligned}\|P_\varepsilon(v)\|_q &\geq \|u^i_\varepsilon\|_q - \|P_\varepsilon(u^i) - u^i_\varepsilon\|_q - \|P_\varepsilon(v) - P_\varepsilon(u^i)\|_q \\ &\geq \|u^i_\varepsilon\|_q - \|u^i - u^i_\varepsilon\|_q - \frac{\delta}{5}.\end{aligned}$$

Since $u^i_\varepsilon \to u^i$ in $L^{p_0}(\Omega)$, thus also in $L^q(\Omega)$, we get that for any ε small enough one has

$$P_\varepsilon(G_0) \ \subset \ C_\varepsilon \setminus B_{L^q(\Omega)}(0, 1-\frac{\delta}{2}).$$

Also notice that the element $P_\varepsilon(G_0)$ of $\mathcal{K}_s(\Omega)$ satisfies $\gamma(P_\varepsilon(G_0)) \geq k$. Then consider the functional $\varphi_\varepsilon : P_\varepsilon(G_0) \to W^{1,p_\varepsilon}_0(\Omega)$ given by $\varphi_\varepsilon(v) := \frac{v}{\|v\|_{p_\varepsilon}}$ and set

$$\forall \varepsilon \in]0, \varepsilon_0[, \qquad G_\varepsilon \ := \ \varphi_\varepsilon(P_\varepsilon(G_0)).$$

Since φ_ε is continuous on $P_\varepsilon(G_0)$, G_ε belongs to $\mathcal{G}^k_{p_\varepsilon}(\Omega)$. Moreover for $\varepsilon > 0$ small enough one has $p_\varepsilon > q$ so that

$$\forall v \in P_\varepsilon(G_0) \qquad 1 - \frac{\delta}{2} \ \leq \ \|v\|_q \ \leq \ \|v\|_{p_\varepsilon} |\Omega|^{\frac{1}{q} - \frac{1}{p_\varepsilon}}.$$

As a consequence one gets

$$\begin{aligned} J^k_\varepsilon(G_\varepsilon) \ &= \ \sup\left\{ F_\varepsilon\left(\frac{v}{\|v\|_{p_\varepsilon}}\right) \ : \ v \in P_\varepsilon(G_0)\right\} \\ &\leq \ \frac{|\Omega|^{\frac{1}{q}-\frac{1}{p_\varepsilon}}}{1-\frac{\delta}{2}} \sup\left\{F_\varepsilon\left(v\right) \ : \ v \in P_\varepsilon(G_0)\right\} \\ &\leq \ \frac{2|\Omega|^{\frac{1}{q}-\frac{1}{p_\varepsilon}}}{2-\delta} \sup\left\{F_\varepsilon\left(v\right) \ : \ v \in C_\varepsilon\right\} \ = \ \frac{2|\Omega|^{\frac{1}{q}-\frac{1}{p_\varepsilon}}}{2-2\delta} \max_{1\leq i \leq m} \left\{F_\varepsilon\left(u^i_\varepsilon\right)\right\}. \end{aligned}$$

As a consequence we have

$$\begin{aligned} \limsup_{\varepsilon \to 0} \left(\inf_{G \in \mathcal{K}_s(\Omega)} J^k_\varepsilon(G)\right) \ &\leq \ \limsup_{\varepsilon \to 0} J_\varepsilon(G_\varepsilon) \\ &\leq \ \frac{2|\Omega|^{\frac{1}{q}-\frac{1}{p_0}}}{2-\delta} \left(\inf_{G \in \mathcal{K}_s(\Omega)} \left\{ J^k_0(G)\right\} + \delta \right). \end{aligned}$$

The conclusion of step 3 then follows by letting δ go to 0 and q go to p_0.

For the case $p_0 = 1$, one has to slightly modify the above argument. We just notice that G_0 being bounded in $W^{1,1}_0(\Omega)$, it is in fact compact in $L^{\frac{2N}{2N-1}}(\Omega)$. We can then proceed as if $p_0 = \frac{2N}{2N-1}$ and follow the same arguments. Notice that this works because if

$$u^i_\varepsilon \to u^i \text{ in } L^1(\Omega) \ \text{ and } \ F_\varepsilon(u^i_\varepsilon) \to F_0(u^i) \ \text{ as } \ \varepsilon \to 0$$

and if the family $(F_\varepsilon)_\varepsilon$ satisfies the uniform growth condition (A2), one infers that $(u^i_\varepsilon)_\varepsilon$ is bounded in $W^{1,1}(\Omega)$ so that

$$u^i_\varepsilon \to u^i \text{ in } L^{\frac{2N}{2N-1}}(\Omega).$$

□

Remark 3.4. The argument used in the third step of the above proof is an adaptation of the proof of Lemma 4.3 in [23]. Notice that the approximating family $(G_\varepsilon)_{\varepsilon>0}$ does not converge to G_0 in $\mathcal{K}_s(\Omega)$, but to a set that can be somewhat larger: this is due to the convexification procedure, which on the one hand ensures that the genus does not decrease, but on the other hand enlarge the approximating sets.

Remark 3.5. The proof of Theorem 3.3 presented above allows to handle in a unified way a wide variety of asymptotic problems, namely all of those covered by the hypotheses $(A1-3)$, such as those of the two following sections. We only notice that the case $p_0 = 1$ requires a specific treatment in Step 3 to handle the fact that the projection in the L^1-norm may be multivalued since this norm is not strictly convex.

We notice that in the third step of the above proof, for any positive ε the set G_ε constructed above belongs to $\Sigma^k_{p_\varepsilon}$: indeed, it is finite dimensional, and thus compact in $W_0^{1,p_\varepsilon}(\Omega)$. As a consequence, if Theorem 3.3 is applied to a constant family $F_\varepsilon := F_0$ for any $\varepsilon > 0$, one obtains the following result.

Corollary 3.6. *Let $p \in [1, +\infty]$, and assume that a functional $F : L^p(\Omega) \to [0, +\infty]$ is convex, 1-homogeneous and satisfies*

$$\begin{cases} \alpha\|\nabla v\|_p \le F(v) \le \beta\|\nabla v\|_p & \text{if } v \in W_0^{1,p}(\Omega), \\ F(v) = +\infty & \text{otherwise}; \end{cases}$$

for some constants $\beta > \alpha > 0$. Then for any positive integer k one has

$$\inf_{G\in\Sigma^k_p} \sup_{v\in G} F(v) = \min_{G\in\mathcal{G}^k_p} \sup_{v\in G} F(v).$$

Proof. We just need to justify that the minimum on $\mathcal{G}^k_p$ is attained. To this end we notice that the functional J^k associated with F is coercive on $\mathcal{K}_s(\Omega)$ as an application of Theorem 3.3, and the Γ-liminf estimate of this Theorem also yields that J^k is lower than its lower semi-continuous envelope, so that it is l.s.c. on $\mathcal{K}_s(\Omega)$. This functional thus attains its minimum on $\mathcal{K}_s(\Omega)$. $\square$

4. Limit as $p \to \infty$ of the eigenvalues of the p-Laplacian

In this section we will apply the results of Section 3 to the approximation of the so called ∞-eigenvalue problem [20]. We first fix some notations, and then consider some consequences of the approximation results.

As we will consider the limit as $p \to \infty$ of the k-th nonlinear eigenvalue of the p-Laplacian defined by (2.3), throughout this section k will denote a fixed positive integer. Without loss of generality, we also assume that $p \ge N+1$ in the following. Since $W_0^{1,N+1}(\Omega)$ is compactly embedded in $\mathcal{C}_0(\Omega)$, we shall also assume that we work with the continuous representative of every function $v \in W_0^{1,p}(\Omega)$. Finally, we denote by $\mathcal{K}_s(\Omega)$ the set whose elements are the compact symmetric subsets of $\mathcal{C}_0(\Omega)$.

For $p \in [N+1, +\infty]$, we now define $J^k_p : \mathcal{K}_s(\Omega) \to [0, +\infty]$ by

$$J^k_p(G) = \begin{cases} \sup_{u\in G} \|\nabla u\|_p & \text{if } G \in \mathcal{G}^p_k(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

As explained above, any element $G \in \mathcal{G}^p_k(\Omega)$ may be considered as an element of $\mathcal{K}_s(\Omega)$ so that one may rewrite (2.3) for $p \in [N+1, +\infty)$ as

$$(\lambda^k_p)^{\frac{1}{p}} = \min\{J^k_p(G) \, : \, G \in \mathcal{K}_s(\Omega)\} \tag{4.1}$$

where the minimum is attained thanks to Corollary 3.6. In a more detailed form, (4.1) reads

$$\begin{aligned} (\lambda^k_p)^{\frac{1}{p}} = \min\{&\sup_{v\in G} \|\nabla v\|_p \, : \, G \in \mathcal{K}_s(\Omega), \\ &G \subset \mathrm{W}_0^{1,p}(\Omega) \cap \{v : \|v\|_p = 1\}, \gamma(G) \ge k\}. \end{aligned}$$

We shall study the Γ-convergence of the family $(J_p^k)_p$ in the space $(\mathcal{K}_s(\Omega), d_{\mathcal{H}})$: in what follows, the distance $d_{\mathcal{H}}$ will denote that induced on $\mathcal{K}_s(\Omega)$ by the sup norm $\|.\|_\infty$ on $\mathcal{C}_0(\Omega)$.

Notice that in the setting of the previous section, the family $(J_p^k)_{p\geq N+1}$ is associated with the family $(F_\varepsilon)_{\varepsilon\leq\frac{1}{N+1}}$ where F_ε is given on $L^1(\Omega)$ by

$$F_\varepsilon(v) := \begin{cases} \|\nabla v\|_{\frac{1}{\varepsilon}} & \text{if } v \in W_0^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

As a consequence of the proof of Lemma 3.2 we know that the genus of $G \in \mathcal{G}_k^p(\Omega)$ as a subset of $W_0^{1,p}(\Omega)$ is the same as its genus as a subset of $\mathcal{C}_0(\Omega)$, so that from now on, we shall always assume that the genus is that computed in $\mathcal{C}_0(\Omega)$. It also follows from Theorem 3.3 that the family $(J_p^k)_{p\in[N+1,+\infty]}$ is equicoercive on $(\mathcal{K}_s(\Omega), d_{\mathcal{H}})$.

We now study the Γ-limit of the family $(J_p^k)_{p\geq N+1}$ as $p\to\infty$. In this case we are able to prove the full Γ-convergence of this family to the functional J_∞^k.

Theorem 4.1. *Let k be a positive integer. Then the family of functionals $(J_p^k)_{p\geq N+1}$ Γ-converges to J_∞^k in $\mathcal{K}_s(\Omega)$ as $p\to+\infty$.*

Proof. The $\Gamma-\liminf$ inequality follows from Theorem 3.3. We thus turn to the proof of the $\Gamma-\limsup$ inequality: let $G_\infty \in \mathcal{K}_s(\Omega)$ be such that $J_\infty^k(G_\infty) < +\infty$, we have to define a family $G_p \overset{d_{\mathcal{H}}}{\to} G_\infty$ such that

$$\limsup_{p\to\infty} J_p^k(G_p) \;\leq\; J_\infty^k(G_\infty).$$

Since G_∞ is a compact subset of $\{v : \|v\|_\infty = 1\}$ and the L^{N+1}-norm is continuous with respect to the L^∞-norm we infer

$$0 < m := \min\{\|u\|_{N+1} : u \in G_\infty\} \leq \max\{|\Omega|, 1\} \min\{\|u\|_p : u \in G_\infty\}$$

for any $p \geq N+1$. Then the application $\varphi_p : G_\infty \to \mathcal{C}_0(\Omega)$ given by $\varphi_p(u) = \frac{u}{\|u\|_p}$ is well defined, bijective and continuous on G_∞. For $p\geq N+1$ we set $G_p = \varphi_p(G_\infty)$. Since G_∞ is compact in $\mathcal{C}_0(\Omega)$, so is G_p in $L^p(\Omega)$ and we conclude that $\gamma(G_\infty) = \gamma(G_p)$. As a consequence, $G_p \in \mathcal{G}_p^k(\Omega)$ and we have

$$\forall p \geq N+1 \qquad J_p^k(G_p) \leq \frac{|\Omega|^{\frac{1}{p}}}{\min\{\|u\|_p : u \in G_\infty\}} J_\infty^k(G_\infty). \tag{4.2}$$

Taking the $\limsup$ as $p\to\infty$ one gets

$$\limsup_{p\to\infty} J_p^k(G_p) \leq J_\infty^k(G_\infty).$$

It remains to prove that $G_p \overset{d_{\mathcal{H}}}{\to} G_\infty$. We infer from the equicoercivity of J_p^k and the previous argument that the family $(G_p)_{p>N+1}$ is precompact in $\mathcal{K}_s$. It is easily seen that any cluster point of $(G_p)_p$ as $p\to+\infty$ contains G_∞. Let $(u_p)_p$ be such that $u_p \in G_p$ for any p and assume that a subsequence $(u_{p_n})_n$ converges to some $u \in \mathcal{C}_0(\Omega)$. For any n there exists $v_n \in G_\infty$ such that $u_{p_n} = \varphi_{p_n}(v_n)$. Since G_∞ is a compact subset of $\mathcal{C}_0(\Omega)$, we can assume without loss of generality that $(v_n)_n$ converges uniformly to some $v \in G_\infty$, but then

$$\|v_n\|_{p_n} \;\to\; \|v\|_\infty \;=\; 1$$

and it follows that $u = v$ belongs to G_∞, so that any cluster point of $(G_p)_p$ in $\mathcal{K}_s$ is included in G_∞, which concludes the proof. $\square$

Remark 4.2. It results from Theorem 4.1 that the functional J^k_∞ is coercive and lower semi-continuous on $\mathcal{K}_s(\Omega)$.

As a consequence of the previous Theorem and the basic properties of the Γ-convergence, we get the following convergence result for the k-th eigenvalues.

Theorem 4.3. *Let λ_p^k be the k-th eigenvalue of the p-Laplacian operator, then*

(1) $\lim_{p\to\infty} (\lambda_p^k)^{\frac{1}{p}} = \Lambda^k_\infty$ *where*

$$\begin{aligned}\Lambda^k_\infty &:= \min_{G\in\mathcal{G}^\infty_k(\Omega)} \sup_{u\in G} \|\nabla u\|_\infty \\ &= \min\Big\{ \sup_{u\in G} \|\nabla u\|_\infty \,:\, G\in\mathcal{K}_s(\Omega), \\ &\qquad G\subset W^{1,\infty}_0(\Omega)\cap\{v : \|v\|_\infty = 1\}, \gamma(G)\geq k\Big\}.\end{aligned}$$

(2) *Let $(G_p)_p$ be a family in $\mathcal{K}_s(\Omega)$ such that $G_p \in \mathcal{G}^k_p(\Omega)$ and $J^k_p(G_p) = \min\{J^k_p(G) \,:\, G\in\mathcal{K}_s(\Omega)\}$ for any p. If for some sequence $p_k \to +\infty$ one has $G_{p_k} \overset{d_{\mathcal{H}}}{\to} G_\infty$, then $J^k_\infty(G_\infty) = \Lambda^k_\infty$.*

Remark 4.4. The point (1) also reads:

$$\lim_{p\to\infty} \min_{G\in\mathcal{G}^k_p(\Omega)} \sup_{u\in G} \|\nabla u\|_p = \min_{G\in\mathcal{G}^\infty_k(\Omega)} \sup_{u\in G} \|\nabla u\|_\infty.$$

Moreover, notice that if $G_p \in \mathcal{G}^k_p(\Omega)$ is such that $J^k_p(G_p) = (\lambda^k_p)^{\frac{1}{p}}$ for any p, then it follows from Theorem 4.1 and 4.3(1) that the family $(G_p)_{p\geq N+1}$ is precompact in $\mathcal{K}_s(\Omega)$.

Proof. We first notice that the family $((\lambda^k_p)^{\frac{1}{p}})_p$ is bounded as $p\to+\infty$. Let $G_\infty \in \mathcal{G}^k_\infty$ be such that $\sup_{u\in G_\infty} \|\nabla u\|_\infty < +\infty$. With the notations of the proof of Theorem 4.1 and using (4.2), we get

$$(\lambda^k_p)^{\frac{1}{p}} = \min\{J^k_p(G) \,:\, G\in\mathcal{K}_s(\Omega)\} \leq J^k_p(\varphi_p(G_\infty)) \leq \frac{\max\{|\Omega|,1\}}{m} J^k_\infty(G_\infty)$$

for any $p\geq N+1$. Since the right hand side does not depend on p, this proves the claim.

As a consequence, the family $(\min_{\mathcal{K}_s} J^k_p)_p$ is bounded, so that (1) and (2) are straightforward consequences of Theorems 2.1 and 4.1. $\square$

We also get the following result for the corresponding k-th eigenfunctions

Theorem 4.5. *For any $p\geq N+1$, let $u_p \in W^{1,p}_0(\Omega)$ be a distributional solution of*

$$-div(|\nabla u_p|^{p-2}\nabla u_p) = \lambda^k_p u_p|u_p|^{p-2} \qquad in\ \Omega.$$

Assume that $\|u_p\|_p = 1$ for any $p\geq N+1$, and that (up to subsequences) $(u_p)_p$ converges in $\mathcal{C}_0(\Omega)$ to some $u_\infty \in W^{1,\infty}_0(\Omega)$ as $p\to\infty$. Then there exists $G_\infty \in \mathcal{K}_s(\Omega)$ such that

$$u_\infty \in G_\infty \qquad and \qquad \|\nabla u_\infty\|_\infty = J^k_\infty(G_\infty) = \Lambda^k_\infty.$$

Remark 4.6. It follows from Lemma 5.2 in [20] that the cluster point u_∞ is a viscosity solution of

$$\begin{cases} \min\{|\nabla u| - \Lambda_\infty^k u\,,\, -\Delta_\infty u\} = 0 & in\ \{u > 0\}, \\ -\Delta_\infty u = 0 & in\ \{u = 0\}, \\ \max\{-|\nabla u| - \Lambda_\infty^k u\,,\, -\Delta_\infty u\} = 0 & in\ \{u < 0\}, \end{cases} \tag{4.3}$$

and by the definition given in [20] this means that u_∞ is an eigenfunction of the infinite-Laplacian for the $\infty-$eigenvalue Λ_∞^k.

Proof. The key point is to show that $\|\nabla u_\infty\|_\infty = \Lambda_\infty^k$. To this end, we notice that $\|u_\infty\|_\infty = 1$ and assume without loss of generality that $u_\infty(x_0) = 1$ for some $x_0 \in \Omega$.

Since $\|\nabla u_p\|_p = (\lambda_p^k)^{\frac{1}{p}} \to \Lambda_\infty^k$, we infer that $\|\nabla u_\infty\|_\infty \le \Lambda_\infty^k$. We show by contradiction that $\|\nabla u_\infty\|_\infty \ge \Lambda_\infty^k$: otherwise, one would have $\|\nabla u_\infty\|_\infty < L < \Lambda_\infty^k$ for some $L > 0$. Let $\varphi \in W^{1,\infty}_{loc}(\mathbb{R}^N)$ be defined by $\varphi(x) = -L|x - x_0|$, then $u_\infty - \varphi$ attains a strict minimum on Ω at x_0. For any $\delta > 0$, consider $\phi_\delta := \rho_\delta * \varphi$, where ρ_δ is a mollifier, i.e. $\rho_\delta(x) := \delta^N \rho(\frac{x}{\delta})$ for some function ρ such that

$$\rho \in C^\infty(\mathbb{R}^N, [0, +\infty[)\,,\ spt(\rho) \subset B(0,1) \quad \text{and} \quad \int_{\mathbb{R}^N} \rho(x)dx = 1\,.$$

As $\varphi_\delta \to \varphi$ uniformly on $\overline{\Omega}$ as $\delta \to 0$, we infer that $u - \varphi_\delta$ attains a local minimum on Ω at some x_δ for δ small enough, and that $x_\delta \to x_0$ as $\delta \to 0$. Since u_∞ is a viscosity solution of (4.3), we conclude that $|\nabla \varphi_\delta(x_\delta)| \ge \Lambda_\infty^k u(x_\delta)$ for δ small enough. It remains to notice that

$$|\nabla \varphi(x_\delta)| \;=\; \left| \int_{\mathbb{R}^N} \rho(y) \nabla \varphi(x_\delta - y) dx \right| \;\le\; L$$

and $u(x_\delta) \to 1$ to obtain the contradiction $L \ge \Lambda_\infty^k$ by letting δ go to 0.

Now, let $F_\infty \in \mathcal{K}_s(\Omega)$ be such that $J_\infty^k(F_\infty) = \Lambda_\infty^k$, and set $G_\infty := F_\infty \cup \{\pm u_\infty\}$, then G_∞ satisfies the desired property. □

Remark 4.7. Notice that for a given $G \in \mathcal{K}_s(\Omega)$ with $J_\infty^k(G) = \Lambda_\infty^k$, a function $u \in G$ such that $\|\nabla u\|_\infty = \Lambda_\infty^k$ is not necessarily a viscosity solution of (4.3). Indeed any function $v \in W_0^{1,\infty}(\Omega)$ with $\|\nabla v\|_\infty = \Lambda_\infty^k$ and $\|v\|_\infty = 1$ may be "added" to such a set G by considering $G \cup \{\pm v\}$, but in general such a function can't be expected to be a solution of (4.3). The above Theorem asserts that the limits of critical points of the Rayleigh quotients are also in the viscosity sense critical for the limit problem.

5. Homogenization of nonlinear eigenvalue problems for p-Laplacian type operators

In the following, p is a fixed real number in $[1, +\infty[$ and k a positive integer. Let f_{hom} and the family $(f_\varepsilon)_{\varepsilon > 0}$ be integrands on $\Omega \times \mathbb{R}^N$ satisfying the assumptions (a1) to (a5) of §2.4 for some common positive constants α, β. For any $\varepsilon > 0$, we define the functional $F_\varepsilon : L^p(\Omega) \to [0, +\infty]$ by

$$F_\varepsilon(v) := \begin{cases} \displaystyle\int_\Omega f_\varepsilon(x, \nabla v(x))dx & \text{if } v \in W_0^{1,p}(\Omega), \\ +\infty & \text{elsewhere.} \end{cases}$$

For any $\varepsilon > 0$ we consider the k-th eigenvalue problem for F_ε, which can be written thanks to Corollary 3.6 in the following way

$$\lambda_\varepsilon^k \;:=\; \min_{G\in\mathcal{G}_k^p(\Omega)} \; \sup_{u\in G} \int_\Omega f_\varepsilon(x,\nabla u)dx \;:=\; \min_{G\in\mathcal{G}_k^p(\Omega)} \; \sup_{u\in G} F_\varepsilon(u).$$

We are interested in the convergence of the family $(\lambda_\varepsilon^k)_{\varepsilon>0}$ as $\varepsilon\to 0$. Following section §3, we shall denote by $\mathcal{K}_s(\Omega)$ be the set whose elements are the compact symmetric subsets of $L^p(\Omega)$, equipped with the Hausdorff distance $d_{\mathcal{H}}$ induced on by the norm $\|.\|_p$. We also define $J_\varepsilon^k : \mathcal{K}_s(\Omega)\to[0,+\infty]$ as follows:

$$J_\varepsilon^k(G) \;:=\; \begin{cases} \sup\{F_\varepsilon(u)^{\frac{1}{p}} \;:\; u\in G\} & \text{if } G\in\mathcal{G}_p^k(\Omega),\\ +\infty & \text{otherwise.}\end{cases}$$

The notations F_{hom}, λ_{hom}^k and J_{hom}^k are defined in the same way as above. Since any element $G\in\mathcal{G}_k^p(\Omega)$ may be considered as an element of $\mathcal{K}_s(\Omega)$ one has

$$\forall\varepsilon>0 \qquad \lambda_\varepsilon^k \;=\; \min\{J_\varepsilon^k(G)^p \;:\; G\in\mathcal{K}_s(\Omega)\}.$$

As a direct consequence of Theorem 3.3 we then get the following convergence result.

Theorem 5.1. *Let the above hypotheses hold, and assume that the family of functionals $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges in $L^p(\Omega)$ to F_{hom} as $\varepsilon\to 0$, then for any positive integer k one has*

$$\lambda_\varepsilon^k \;\to\; \lambda_{hom}^k \;=\; \min_{G\in\mathcal{G}_k^p(\Omega)} \; \sup_{u\in G} F_{hom}(u)$$

as ε goes to 0.

In [5] the authors consider the homogenization for non-linear eigenvalue problems related to a family of monotone elliptic operators of the form $\mathcal{A}_\varepsilon := -div(a_\varepsilon(\cdot,\cdot))$ which $G-$converge to an operator $\mathcal{A}_{hom}$ of the same form. Under the assumptions (h1-5) the operators $\mathcal{A}_\varepsilon$ are the sub-differentials of integral functionals of the type F_ε whose integrands satisfy (a1-5), see §2.4. Theorem 5.1 then implies that for any positive integer k the generalized sequence of the k-th eigenvalues (as defined in section §2.4) of the operators $\mathcal{A}_\varepsilon$ considered in [5] converges to the k-th eigenvalue of the limit operator.

Remark 5.2. Theorem 3.3 also yields that $\Gamma - \liminf J_\varepsilon^k \geq J_{hom}^k$ on $\mathcal{K}_s(\Omega)$, but unlike in the previous section one can't expect the family $(J_\varepsilon^k)_\varepsilon$ to Γ-converge to J_{hom}^k. Indeed, in Theorem 5.1 it is only assumed that $(F_\varepsilon)_{\varepsilon>0}$ Γ-converges in $L^p(\Omega)$ to F_{hom}, whereas in section §4 the functionals $F_\varepsilon(v) = \|\nabla v\|_{\frac{1}{\varepsilon}}$ not only Γ-converge but also pointwise converge to $F_0(v) = \|\nabla v\|_\infty$ on $\mathcal{C}_0(\Omega)$, which allows to obtain the Γ-limsup estimate for $(J_\varepsilon^k)_\varepsilon$. Of course the proof of Theorem 4.1 could be adapted to get that $(J_\varepsilon^k)_\varepsilon$ Γ-converges to J_{hom}^k when $(F_\varepsilon)_{\varepsilon>0}$ also converges pointwise, but this last hypothesis is not usual in the context of homogenization.

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References

[1] Ambrosio, L., Tilli, P., Topics on Analysis in Metric Spaces, Oxford Lecture Series in Mathematics and Its Applications, **25**, Oxford University Press, Oxford, 2004.

[2] Appell, J., De Pascale, E., Vignoli, A. Nonlinear spectral theory, de Gruyter Series in Nonlinear Analysis and Applications **10**. Berlin (2004).

[3] Aronsson, G., Crandall, M. G., Juutinen P., A tour of the theory of absolutely minimizing functions. Bull. Amer. Math. Soc. **41** (2004), 439–505.

[4] Attouch, H., Variational convergence for functions and operators, Pitman, Boston (1984).

[5] Baffico, L., Conca, C., Rajesh, M., Homogenization of a class of non-linear eigenvalue problems, Proc. Roy. Soc. Edinburgh Sect. A, **136** (2006) , no.1, 7–22.

[6] Belloni, M., Kawohl, B., The pseudo-p-Laplace eigenvalue problem and viscosity solutions as $p \to \infty$, ESAIM. Control, Optimisation and Calculus of Variations **10** (2004), 28–52.

[7] Belloni, M., Kawohl, B., A direct uniqueness proof for equations involving the p-Laplace operator, Manuscripta Math. **109** (2002), no. 2, 229–231.

[8] Browder, F. E., Lusternik-Schnirelman category and nonlinear elliptic eigenvalue problems, Bull. Am. Math. Soc. **71** (1965), 644–648.

[9] Buttazzo, G., Dal Maso, G., An existence result for a class of shape optimization problems, Arch. Rational Mech. Anal. **122** (1993), no. 2, 183–195.

[10] Champion, T., De Pascale, L., Homogenization of Dirichlet problems with convex bounded constraints on the gradient. Z. Anal. Anwendungen **22** (2003), no. 3, 591–608.

[11] Champion, T., De Pascale, L., Prinari, F., Γ-convergence and absolute minimizers for supremal functional. ESAIM. Control, Optimisation and Calculus of Variations **10** (2004), 14–27.

[12] Chiadó Piat, V., Dal Maso, G., Defranceschi, A., G-Convergence of monotone operators, Ann. Inst. Henry Poincaré **7** (1990), no. 3, 123–160.

[13] Coffman, C. V., Lyusternik-Schnirelman theory and eigenvalue problems for monotone potential operators, J. Funct. Anal. **14** (1973), 237–252.

[14] De Giorgi, E., Franzoni, T., Su un tipo di convergenza variazionale, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), no.6, 842–850.

[15] Dal Maso, G., An Introduction to $\Gamma-$Convergence, Progress in Nonlinear Differential Equations and their Applications. **8**. Birkhauser, Basel (1993).

[16] Dunford, N., Schwartz, J. T., Linear Operators Part I: General Theory, Wiley Classics Library Edition (New York) 1988.

[17] Francaviglia, S., Franzoni, T., Γ-limits of functions of two variables, (Appendix to: Franzoni, T., Abstract Γ-convergence), Optimization and related fields. Proceedings of the G. Stampacchia international school of mathematics held at Erice, September 17–30, 1984. Edited by R. Conti, E. De Giorgi and F. Giannessi. Lecture Notes in Mathematics, 1190. Springer-Verlag, Berlin, (1986).

[18] García Azorero, J. P., Peral Alonso, I., Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues, Comm. Partial Differential Equations **12** (1987), 1389–1430.

[19] García Azorero, J. P., Peral Alonso, I., Comportement asymptotique des valeurs propres du p-laplacien. [Asymptotic behavior of the eigenvalues of the p-Laplacian] C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 2, 75–78.

[20] Juutinen, P., Lindqvist, P., On the higher eigenvalues for the $\infty-$eigenvalue problem, Calc. Var. Partial Differential Equations **23** (2005), no. 2, 169–192.

[21] Juutinen, P., Lindqvist, P., Manfredi, J. J., The ∞-eigenvalue problem, Arch. Ration. Mech. Anal. **148** (1999), no. 2, 89–105.

[22] Juutinen, P., Lindqvist, P., Manfredi, J. J., The infinity Laplacian: examples and observations, Papers on analysis, 207–217, Rep. Univ. Jyväskylä Dep. Math. Stat., 83, Univ. Jyväskylä, Jyväskylä, 2001.

[23] Kaizu, S., Nishiyama, A., Watanabe, J., Convergence of solutions of nonlinear eigenvalue problems. Preprint.

[24] Lindqvist, P., On a nonlinear eigenvalue problem. Fall School in Analysis (Jyväskylä, 1994), 33–54, Report, 68, Univ. Jyväskylä, Jyväskylä, 1995.

[25] Mosco, U., Convergence of convex sets and of solutions of variational inequalities, Advances in Math. **3** (1969), 510–585.

[26] Rabinowitz, P. H, Minimax Methods in Critical Point Theory with Applications to Differential Equations, Conference board of the mathematical sciences, regional conference series

in mathematics, **65**, Published by the American Mathematical Society, Providence, Rhode Island 1986.

[27] Struwe, M., Variational Methods, Springer-Verlag, Berlin Heidelberg 1990.

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