

Published in
 Adv. Math. Sci. Appl. 16 (2006), no. 2, pp. 377–392.

A PARTICULAR CLASS OF SOLUTIONS OF A SYSTEM OF EIKONAL EQUATIONS

THIERRY CHAMPION AND GISELLA CROCE

ABSTRACT. In this paper we study the following Dirichlet problem:

$$\begin{cases} \left| \frac{\partial u}{\partial x_i} \right| = 1 & \text{a.e. in } \Omega, i = 1, \dots, N \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^N and $u \in W^{1,\infty}(\Omega)$. This problem has infinitely many solutions: our aim is to isolate those which minimize some functional involving the discontinuity sets of $\frac{\partial u}{\partial x_i}$, for $i = 1, \dots, N$.

1. INTRODUCTION

In this paper we are interested in the following system of eikonal equations

$$\begin{cases} \left| \frac{\partial u}{\partial x_i} \right| = 1 & \text{a.e. in } \Omega, i = 1, \dots, N \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a connected open bounded subset of \mathbb{R}^N with Lipschitz boundary and $u \in W^{1,\infty}(\Omega) \cap C_0(\Omega)$. This problem admits infinitely many solutions (see [6] and [9] for example for a proof). An interesting question is then to isolate a particular class of solutions which have some additional properties, since such an analysis could be useful to select and characterize a preferred solution.

There exist some evident difficulties to this kind of question. For example the lack of convexity of the set $\mathfrak{S}(\Omega)$ of the solutions of (1.1) implies that the natural functionals

$$v \rightarrow \int_{\Omega} |v|^p \quad p \geq 1,$$

have in general neither a minimizer nor a maximizer over $\mathfrak{S}(\Omega)$. Indeed any minimizing sequence converges to 0, which does not belong to $\mathfrak{S}(\Omega)$, and the cluster points of maximizing sequences are the distance function to $\partial\Omega$ (in the l^1 norm of \mathbb{R}^N) or its opposite, which usually do not verify (1.1) (see remark 3.6).

Another natural way to select a particular solution in $\mathfrak{S}(\Omega)$ would be to use the notion of viscosity solution (for the definition and further details see [9]). Indeed one could study the existence of $W^{1,\infty}(\Omega)$ viscosity solutions of a problem like

$$\begin{cases} F(Du) = 0 & \text{a.e. in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1.2)$$

where $F : \mathbb{R}^N \rightarrow \mathbb{R}$ is such that $F(x) = 0$ if and only if $x = (x_1, \dots, x_N)$ verifies $|x_i| = 1 \forall i = 1, \dots, x_N$. This is indeed possible in dimension $N = 1$, where the

only viscosity solution of the problem

$$\begin{cases} |u'| - 1 = 0 & \text{a.e. in }]-1, 1[\\ u(-1) = u(1) = 0 \end{cases} \quad (1.3)$$

is $u(x) = 1 - |x|$. Nevertheless when $N \geq 2$ there is in general no viscosity solution of (1.2), unless Ω is convex and the normals to $\partial\Omega$ are in the set $E = \{x : |x_i| = 1/\sqrt{N}\}$; we refer to [5] and [13] for further details.

In the literature we find few articles with the purpose of isolating particular solutions of problem (1.1) and selecting one. In [7] and [8] the authors study numerically a variational problem over the set of non-negative solutions of (1.1) in the 2-dimensional case: they obtain a maximizing sequence for the problem

$$\sup \left\{ \int_{\Omega} u, \quad u \geq 0, \quad u \in \mathfrak{S}(\Omega) \right\} \quad (1.4)$$

through the numerical minimization of the functional

$$u \mapsto - \int_{\Omega} u + \frac{1}{2} \int |\nabla u|^2 + \frac{\varepsilon}{2} \int |\Delta u|^2 + \frac{1}{2\varepsilon} \sum_{i=1}^N \int \left(\left| \frac{\partial u}{\partial x_i} \right| - 1 \right)^2.$$

Unfortunately, as we said above, there is in general no optimal solution for the variational problem (1.4), since the limit for any maximizing sequence is the distance to $\partial\Omega$ (in the l^1 norm).

A different approach is followed in [10] where the authors construct an explicit solution. Indeed they define a special partition of Ω made up of convex sets whose normals to $\partial\Omega$ are in E ; over each of these sets they consider the relative viscosity solution. In this way the constructed solution has local properties (which depend on the chosen partition), but no global ones. In any case we want to point out that this study is made in a more general framework than problem (1.1).

Our purpose in this work is to characterize a particular class of solutions of (1.1) through a variational method, i.e. as the optimal solutions of a variational problem. We shall define over $\mathfrak{S}(\Omega)$ a functional involving the discontinuity sets of the gradient of the solutions of (1.1) and minimize it. Let us briefly explain the kind of results that we shall present in this article. We distinguish two cases, according to the properties of Ω . In the case where $\partial\Omega$ is composed by a finite set of faces with normals in E (see section 3), we consider the functional

$$v \mapsto \sum_{i=1}^N \mathcal{H}^{N-1}(J_{\frac{\partial v}{\partial x_i}})$$

(where $J_{\frac{\partial v}{\partial x_i}}$ is the jump set of $\frac{\partial v}{\partial x_i}$) and we show that it admits a minimizer.

In the general case, that is when Ω is any connected open subset of \mathbb{R}^N with Lipschitz boundary, we cannot minimize the discontinuity sets of Du for $u \in \mathfrak{S}(\Omega)$ in the same way. Indeed the previous functional is in general infinite over $\mathfrak{S}(\Omega)$, as shown in section 4. Consequently we construct a positive “weight” function $h \in C_0(\Omega)$ such that the functional

$$v \rightarrow \sum_{i=1}^N \int_{J_{\frac{\partial v}{\partial x_i}}} h(x) d\mathcal{H}^{N-1}(x)$$

admits a minimizer over $\mathfrak{S}(\Omega)$.

We also study the relationship between the viscosity solution (when it exists) and the optimal solutions of the minimization problem: we show that they coincide if $N = 1, 2$ (see Proposition 3.8).

Our study allows us to isolate particular solutions of problem (1.1), but unfortunately we are not able to say if it selects a particular one, in the sense that the variational problems we propose could admit more than one solution. In section 5 we then present a method to select a function among the optimal solutions of the variational problems that we studied (see 4.).

Finally we refer to a recent work [3] which does not address the problem of the selection of a particular solution of (1.1) but is also concerned with some local properties of the discontinuity sets of $Du, u \in \mathfrak{S}(\Omega)$.

2. NOTATIONS

Throughout this paper Ω denotes a bounded connected open subset of \mathbb{R}^N with Lipschitz boundary. We will denote by $(x_i)_{1 \leq i \leq N}$ the coordinates of a point $x \in \mathbb{R}^N$, and $\langle x, y \rangle$ the usual scalar product of $x, y \in \mathbb{R}^N$. The norm $\|x\|_{l^1}$ of $x \in \mathbb{R}^N$ stands for the usual l^1 norm, that is $\|x\|_{l^1} = \sum_{i=1}^N |x_i|$.

In the following, we shall use the short notation $W_0^{1,\infty}(\Omega)$ for $W^{1,\infty}(\Omega) \cap C_0(\Omega)$. When considering a function $v \in W_0^{1,\infty}(\Omega)$, we shall always assume that we are handling its continuous representative; notice that, due to the hypothesis on Ω , such a function v is a Lipschitz continuous function.

Let us briefly recall some properties and notations of the spaces BV and SBV for future use (in any case we refer to [2] and to [11] for further details). If $w \in BV(\Omega)$, we write $\|w\|_{BV(\Omega)}$ for the usual BV norm of w defined by

$$\|w\|_{BV(\Omega)} = \|w\|_{L^1(\Omega)} + |Dw|(\Omega).$$

Moreover we will use the usual weak* convergence in BV defined as follows (see [2]):

Definition 2.1 (weak* convergence in BV). Let $(u_n)_n, u \in BV(\Omega)$. We say that $(u_n)_n$ weakly* converges to u in $BV(\Omega)$ if $u_n \rightarrow u$ in $L^1(\Omega)$ and the measures Du_n weakly* converge to the measure Du in $\mathcal{M}(\Omega, \mathbb{R}^N)$, that is

$$\lim_{n \rightarrow \infty} \int_{\Omega} \varphi dDu_n = \int_{\Omega} \varphi dDu \quad \forall \varphi \in C_0(\Omega).$$

A BV function belongs to SBV when the Cantor part of its derivative is zero. This implies that the measure Dw of an $SBV(\Omega)$ function can be decomposed in the following way:

$$\begin{aligned} Dw &= D^a w + D^j w \\ &= \nabla w \mathcal{L}^N + (w^+ - w^-) \nu_w \mathcal{H}^{N-1} \llcorner J_w = \nabla w \mathcal{L}^N + [w] \nu_w \mathcal{H}^{N-1} \llcorner J_w \end{aligned}$$

where $D^a w$ is the absolute continuous part of Dw with respect to the Lebesgue measure \mathcal{L}^N in \mathbb{R}^N , $D^j w = [w] \nu_w \mathcal{H}^{N-1} \llcorner J_w$ is the *jump* part of Dw , \mathcal{H}^{N-1} the Hausdorff measure of dimension $N - 1$, w^+ and w^- denote the upper and lower approximate limits of w , J_w the jump set of w and ν_w its generalized normal.

Referring to our problem, we will denote by $\mathfrak{S}(\Omega)$ the set of $W^{1,\infty}(\Omega)$ solutions of (1.1), that is

$$\mathfrak{S}(\Omega) := \left\{ v \in W_0^{1,\infty}(\Omega) : \left| \frac{\partial v}{\partial x_i} \right| = 1 \text{ a.e. in } \Omega, i = 1, \dots, N \right\}.$$

In this paper we will consider the piecewise C^1 elements v of $\mathfrak{S}(\Omega)$ and more generally those elements v of $\mathfrak{S}(\Omega)$ such that $Dv \in SBV_{loc}(\Omega)^N$. To define a piecewise C^1 function, we first recall the definition of Caccioppoli partitions and piecewise constant function (we refer to [2] for further details):

Definition 2.2 (Caccioppoli partition). Let $I \subset \mathbb{N}$. A partition $\{E_i\}_{i \in I}$ of Ω is a Caccioppoli partition if $\sum_{i \in I} |D\chi_{E_i}|(\Omega) < \infty$.

In the above definition, χ_A denotes the characteristic function of a set A , that is $\chi_A(x) = 1$ when $x \in A$, $\chi_A(x) = 0$ when $x \notin A$.

Definition 2.3 (Piecewise constant function). We say that $u : \Omega \rightarrow \mathbb{R}$ is piecewise constant in Ω if there exist a Caccioppoli partition $\{E_i\}_{i \in I}$ of Ω and a function $t : I \rightarrow \mathbb{R}$ such that $u = \sum_{i \in I} t_i \chi_{E_i}$.

Notice that a piecewise constant function is an SBV function (see [2] for further details).

We define the set of piecewise C^1 functions by

$$C_{pw}^1(\Omega) := \left\{ v \in W^{1,\infty}(\Omega) : \frac{\partial v}{\partial x_i} \text{ is piecewise constant in } \Omega, i = 1, \dots, N \right\}.$$

As has already been said, we shall handle the solutions of (1.1) such that $Dv \in SBV_{loc}(\Omega)^N$. We now want to write their distributional gradient: for any index $i \in \{1, \dots, N\}$ we use the short notation J_i^v instead of $J_{\frac{\partial v}{\partial x_i}}$. Therefore for a function $v \in \mathfrak{S}(\Omega)$ such that $Dv \in SBV_{loc}(\Omega)^N$ one has

$$D \left(\frac{\partial v}{\partial x_i} \right) = 2\nu_{\frac{\partial v}{\partial x_i}} \mathcal{H}^{N-1} \llcorner J_i^v \quad \text{on } \omega \quad \forall i = 1, \dots, N \quad (2.1)$$

for any open subset $\omega \subset \bar{\omega} \subset \Omega$.

In the rest of the paper, the distance in l^1 norm to a subset of \mathbb{R}^N will play an important role: if A is a bounded subset of \mathbb{R}^N , we define $d_{l^1}(\cdot, A) : \mathbb{R}^N \rightarrow \mathbb{R}$ by

$$\forall x \in \mathbb{R}^N, \quad d_{l^1}(x, A) := \inf \{ \|x - y\|_{l^1} : y \in A \}.$$

3. A PARTICULAR CASE

In this section we assume that Ω satisfies the following property:

$$(H) \quad \left\{ \begin{array}{l} \text{the boundary } \partial\Omega \text{ is the union of a finite family of polyhedrons} \\ \text{of dimension } N-1, \text{ each of which is included in a hyperplane} \\ \text{with normals in } E, \end{array} \right.$$

where $E := \{x \in \mathbb{R}^N : \forall i, |x_i| = 1/\sqrt{N}\}$, and we minimize the \mathcal{H}^{N-1} measure of the discontinuity sets of $\frac{\partial v}{\partial x_i}$, for $v \in \mathfrak{S}(\Omega)$. We define

$$\mathcal{E}_{pw}(\Omega) := \{v \in \mathfrak{S}(\Omega) : v \in C_{pw}^1(\Omega)\}.$$

We now state the main result of this section.

Theorem 3.1. *Let Ω be an open bounded connected subset of \mathbb{R}^N which satisfies hypothesis (H). Then the variational problem*

$$(P_1) \quad \inf \left\{ \mathcal{F}(v) := \sum_{i=1}^N \mathcal{H}^{N-1}(J_i^v) : v \in \mathcal{E}_{pw}(\Omega) \right\}.$$

has finite value and admits an optimal solution.

Remark 3.2. Under assumption (H), problem (P₁) always has at least two optimal solutions: indeed, if u is a solution then $-u$ is also a solution.

Remark 3.3. When Ω fails to satisfy hypothesis (H), problem (P₁) is not well posed as we will show in section 4.

In the proof of Theorem 3.1 we will use the following lemmas.

Lemma 3.4. *Let Ω be an open bounded connected subset of \mathbb{R}^N with Lipschitz boundary. Then*

$$-d_{l^1}(\cdot, \partial\Omega) \leq v \leq d_{l^1}(\cdot, \partial\Omega) \quad \text{on } \overline{\Omega}$$

for every function $v \in \mathfrak{S}(\Omega)$.

Proof. Let $v \in \mathfrak{S}(\Omega)$. We denote by A a set of zero Lebesgue measure such that v is differentiable on $\Omega \setminus A$ with $\left| \frac{\partial v}{\partial x_i} \right| = 1$ for all i .

Take $x \in \Omega$, and let $y \in \partial\Omega$ be such that $d_{l^1}(x, \partial\Omega) = \|x - y\|_{l^1}$. For any $t \in]0, 1[$ we define the map $\gamma_t : [0, 1] \rightarrow \Omega$ by $\gamma_t(s) := sx + (1-s)y$, where $x^t := tx + (1-t)y$. For a fixed $t \in]0, 1[$ and $\varepsilon > 0$, we apply Lemma 3.2 in [4] to get the existence of a Lipschitz continuous path $\gamma_{t,\varepsilon} : [0, 1] \rightarrow \Omega \setminus A$ such that

$$\gamma_{t,\varepsilon}(0) = x^t, \quad \gamma_{t,\varepsilon}(1) = x \quad \text{and} \quad \|\gamma'_{t,\varepsilon} - \gamma'_t\|_{L^\infty(0,1)} \leq \varepsilon.$$

Then the function $v \circ \gamma_{t,\varepsilon}$ is Lipschitz continuous on $[0, 1]$ and we can compute

$$v(x) - v(x^t) = \int_0^1 (v \circ \gamma_{t,\varepsilon})'(s) ds = \int_0^1 \nabla v(\gamma_{t,\varepsilon}(s)) \cdot \gamma'_{t,\varepsilon}(s) ds.$$

Now $\left| \frac{\partial v}{\partial x_i}(\gamma_{t,s}(s)) \right| = 1$ for all i and so

$$v(x) - v(x^t) \leq \int_0^1 \|\gamma'_{t,\varepsilon}(s)\|_{l^1} ds \leq N\varepsilon + \int_0^1 \|\gamma'_t(s)\|_{l^1} ds = N\varepsilon + \|x - x^t\|_{l^1}.$$

Letting t and ε go to 0 this yields $v(x) \leq d_{l^1}(x, \partial\Omega)$. In an analogous way one can prove that $-v(x) \leq d_{l^1}(x, \partial\Omega)$ \square

Lemma 3.5. *Let Ω be an open bounded connected subset of \mathbb{R}^N which satisfies hypothesis (H). Then $d_{l^1}(\cdot, \partial\Omega)$ belongs to $\mathcal{E}_{pw}(\Omega)$ and $\mathcal{F}(d_{l^1}(\cdot, \partial\Omega)) < +\infty$.*

Remark 3.6. Notice that for a general bounded open subset Ω , $d_{l^1}(\cdot, \partial\Omega)$ does not necessarily satisfy $\left| \frac{\partial v}{\partial x_i} \right| = 1$ a.e. in Ω for all i . Indeed, it is sufficient to consider $\Omega = (0, 1)^2 \subset \mathbb{R}^2$: we have that $d_{l^1}((x_1, x_2), \partial\Omega) = x_2$ in the set $T = \{(x_1, x_2) : 0 \leq x_1 \leq 1/2, 0 \leq x_2 \leq x_1\}$.

Proof. Since Ω satisfies hypothesis (H), we can write $\partial\Omega = \bigcup_{j=1}^m F_j$ where each face

F_j is a polyhedron of dimension $N - 1$ included in a hyperplane H_j with normal in E . We shall assume that two faces F_j and F_k that intersect are not included in the same hyperplane of \mathbb{R}^N .

Let $j \in \{1, \dots, m\}$. We first notice that since the normal of the hyperplane H_j is included in E , one has $\left| \frac{\partial}{\partial x_i} d_{l^1}(\cdot, H_j) \right| = 1$ a.e. in Ω for all i . Let us prove that

$d_{l^1}(\cdot, \partial\Omega) = \min_{1 \leq j \leq m} d_{l^1}(\cdot, F_j)$ also satisfies this system of eikonal equations in Ω .

We fix x in Ω and distinguish three cases:

Case 1. Assume that $d_{l^1}(x, \partial\Omega) = d_{l^1}(x, F_j)$ for a unique index j . Then $d_{l^1}(x, F_j)$ is attained in the relative interior of F_j , so that in a neighborhood of x one has $d_{l^1}(\cdot, \partial\Omega) = d_{l^1}(\cdot, H_j)$, and thus in this neighborhood $d_{l^1}(\cdot, \partial\Omega)$ is regular and satisfies the system of eikonal equations (1.1).

Case 2. Assume that $d_{l^1}(x, \partial\Omega) = d_{l^1}(x, F_j) = d_{l^1}(x, F_k)$ is attained in $F_j \cap F_k$ with $j \neq k$. Then $d_{l^1}(x, F_j) = d_{l^1}(x, F_k)$ is attained in the intersection $H_j \cap H_k$. We notice that the set of the elements y of \mathbb{R}^N such that $d_{l^1}(y, F_j) = d_{l^1}(y, F_k)$ is attained in $F_j \cap F_k$ is included in the hyperplane $A_{j,k}$ generated by $H_j \cap H_k$ and $\xi_j + \xi_k$, where ξ_l denotes a normal vector to H_l for $l \in \{j, k\}$. Since $\mathcal{L}^N(A_{j,k}) = 0$ this set is negligible in view of the system of eikonal equations (1.1).

Case 3. Assume that $d_{l^1}(x, \partial\Omega) = d_{l^1}(x, F_j) = d_{l^1}(x, F_k)$ for $j \neq k$ such that the hyperplanes H_j and H_k are parallel and that case 2 is excluded. Then $d_{l^1}(x, F_l)$ is attained in the relative interior of F_l for $l \in \{j, k\}$, and thus one has $d_{l^1}(x, H_j) = d_{l^1}(x, H_k)$. As a consequence, x belongs to the set $B_{j,k} := \{y : d_{l^1}(y, H_j) = d_{l^1}(y, H_k)\}$, which is an hyperplane parallel to H_j and H_k , so that $\mathcal{L}^N(B_{j,k}) = 0$. This means that this set is negligible in view of the system of eikonal equations (1.1).

As a consequence of the above discussion, almost every x in Ω enters case 1, so that $d_{l^1}(\cdot, \partial\Omega)$ is a solution of (1.1). Moreover, if $i \in \{1, \dots, N\}$ then one has

$$\begin{aligned} J_i^{d_{l^1}(\cdot, \partial\Omega)} &= J_i^{d_{l^1}(\cdot, \partial\Omega)} \cap \bigcup_{j,k} (A_{j,k} \cup B_{j,k}) \\ &\subset \bigcup_{j,k} [\Omega \cap (A_{j,k} \cup B_{j,k})]. \end{aligned}$$

Since Ω is bounded and the sets $A_{j,k}$ and $B_{j,k}$ are hyperplanes of \mathbb{R}^N , we infer that the last union has finite \mathcal{H}^{N-1} measure. This being true for all index i , we get that $\mathcal{F}(d_{l^1}(\cdot, \partial\Omega))$ is finite. \square

We can now turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. As a direct consequence of Lemma 3.5, we get that the infimum $\inf(P_1)$ is finite. We now apply the direct method of the Calculus of Variations to prove the existence of an optimal solution of (P_1) .

Let $(v^n)_{n \in \mathbb{N}} \subset \mathcal{E}_{pw}(\Omega)$ be a minimizing sequence for (P_1) . We deduce from Lemma 3.4 that $(v^n)_n$ is uniformly bounded in $L^\infty(\Omega)$. Moreover this sequence is uniformly Lipschitz continuous on $\bar{\Omega}$ since $\partial\Omega$ is Lipschitz and $|\nabla v^n| \leq \sqrt{N}$ a.e. in Ω for any n . Applying Ascoli-Arzelà and Banach-Alaoglu Theorems, and up to a subsequence, we can assume that $v^n \rightarrow v^\infty$ in $C_0(\Omega)$ and $v^n \rightharpoonup v^\infty$ weakly* in $W^{1,\infty}(\Omega)$ for some $v^\infty \in W_0^{1,\infty}(\Omega)$.

Step 1: we show that $v^\infty \in \mathcal{E}_{pw}(\Omega)$. We first prove that $\left| \frac{\partial v^\infty}{\partial x_i} \right| = 1$ a.e. in Ω . Let us fix $i \in \{1, \dots, N\}$. We observe that since $(v^n)_n$ is a minimizing sequence, there exists a constant $C > 0$ such that $\mathcal{F}(v^n) \leq C$ for every $n \in \mathbb{N}$. This implies that

$$\left\| \frac{\partial v^n}{\partial x_i} \right\|_{BV(\Omega)} = \mathcal{L}^N(\Omega) + 2\mathcal{H}^{N-1}(J_i^{v^n}) \leq \mathcal{L}^N(\Omega) + 2C. \quad (3.1)$$

Applying Theorem 6.1 to the sequence $\frac{\partial v^n}{\partial x_i}$ we have that $\frac{\partial v^n}{\partial x_i} \rightharpoonup g_i$ weakly* in $BV(\Omega)$ for some function $g_i \in BV(\Omega)$ (up to a subsequence). Since we already know that $\frac{\partial v^n}{\partial x_i} \rightharpoonup \frac{\partial v^\infty}{\partial x_i}$ weakly* in $L^\infty(\Omega)$, we infer that $\frac{\partial v^\infty}{\partial x_i} = g_i$. Moreover $\frac{\partial v^n}{\partial x_i} \rightarrow \frac{\partial v^\infty}{\partial x_i}$ in $L^1(\Omega)$ so that $|\frac{\partial v^\infty}{\partial x_i}| = 1$ a.e. in Ω .

To prove that v^∞ belongs to $C_{pw}^1(\Omega)$ it is sufficient to apply Theorem 6.3 to the sequence $\frac{\partial v^n}{\partial x_i}$. Indeed, $\frac{\partial v^n}{\partial x_i}$ is a piecewise constant function for every n and $\left\| \frac{\partial v^n}{\partial x_i} \right\|_{L^\infty(\Omega)} + \mathcal{H}^{N-1}(J_i^{v^n})$ is uniformly bounded since $\mathcal{F}(v^n) \leq C$. As a consequence, up to a subsequence, $\frac{\partial v^n}{\partial x_i}$ converges in measure to a piecewise constant function which is necessarily $\frac{\partial v^\infty}{\partial x_i}$. This proves that $v^\infty \in \mathcal{E}_{pw}(\Omega)$.

Step 2: we show that v^∞ is a minimizer of \mathcal{F} . Since $(v_n)_n$ is a minimizing sequence for \mathcal{F} it is sufficient to prove that

$$\liminf_{n \rightarrow \infty} \mathcal{H}^{N-1}(J_i^{v^n}) \geq \mathcal{H}^{N-1}(J_i^{v^\infty}) \quad (3.2)$$

for all $i = 1, \dots, N$. Thanks to (2.1) one has for any $n \in \mathbb{N} \cup \{+\infty\}$

$$\left| D \frac{\partial v^n}{\partial x_i} \right|(\Omega) = 2\mathcal{H}^{N-1}(J_i^{v^n}).$$

Moreover we know that $\frac{\partial v^n}{\partial x_i} \rightharpoonup \frac{\partial v^\infty}{\partial x_i}$ weakly* in $BV(\Omega)$. Applying Theorem 6.4 with $f \equiv 1$ we get (3.2). \square

We now study the relationship between the viscosity solutions of problem (1.1) and the optimal solutions of problem (P_1) . We restrict ourselves to the one and two-dimensional cases, where the geometric arguments are intuitive.

In the one-dimensional case, it is evident that the viscosity solution $u(x) = 1 - |x|$ and its opposite minimize the number of jumps of the solutions of (1.3), as their derivatives have just one jump. For the 2-dimensional case, we recall the following result about the existence of a viscosity solution (see Example 1.3 and Theorem 3.3 of [13]).

Proposition 3.7. *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function such that $F(\xi_1, \xi_2) = 0$ if and only if $|\xi_i| = 1, i = 1, 2$ and $F(\xi_1, \xi_2) < 0$ if $|\xi_i| < 1$ for $i = 1, 2$. Let Ω be an open bounded connected subset of \mathbb{R}^2 . Then there is no $W^{1,\infty}(\Omega)$ viscosity solution of problem*

$$\begin{cases} F(Dv) = 0 & \text{a.e. in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

unless Ω is a rectangle whose normals are in the set $E = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_i| = \frac{1}{\sqrt{2}}, i = 1, 2\}$. In this case the only viscosity solution is $d_{l_1}(\cdot, \partial\Omega)$.

It also happens that $d_{l_1}(\cdot, \partial\Omega)$ and its opposite are the only optimal solutions of (P_1) if Ω is a rectangle with normals in E , as the following result shows. Notice that if $\Omega \subset \mathbb{R}^2$ is convex and satisfies property (H) , then Ω is necessarily such a rectangle.

Proposition 3.8. *Let Ω be a rectangle with normals in E . Then $\pm d_{l_1}(\cdot, \partial\Omega)$ are the only optimal solutions of (P_1) .*

We will use the following lemma in the proof:

Lemma 3.9. *If $P : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the projection $(x_1, x_2) \rightarrow x_2$, and E is a measurable set of \mathbb{R}^2 then*

$$\int_{\mathbb{R}} \mathcal{H}^0(E \cap P^{-1}\{y\}) dy \leq \mathcal{H}^1(E).$$

Proof. It is sufficient to apply Theorem 2.10.25 of [12] with $f = P$, $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $k = 0$ and $m = 1$. \square

We are now able to prove Proposition 3.8:

Proof. To simplify the notations, we set $d := d_{l^1}(\cdot, \partial\Omega)$. Let $v \in \mathcal{E}_{pw}(\Omega)$ different from d and $-d$: we want to prove that

$$\mathcal{F}(v) = \sum_{i=1}^2 \mathcal{H}^1(J_i^v) > \sum_{i=1}^2 \mathcal{H}^1(J_i^d) = \mathcal{F}(d) = \mathcal{F}(-d).$$

We first claim that $\mathcal{H}^1(J_1^v) > \mathcal{H}^1(J_1^d)$. To this end, we define

$$A := \{t \in \mathbb{R} : \exists s \in \mathbb{R} \text{ such that } x := (s, t) \in \Omega \text{ and } -d(x) < v(x) < d(x)\}.$$

Since the functions v , d and $-d$ are continuous, the set A can be written as $\bigcup_{i \in I}]t_i, t'_i[$

where $I \subset \mathbb{N}$ is non empty, and $t_i < t'_i$ for all $i \in I$. We notice that

$$\mathcal{H}^1(J_1^v \cap (\mathbb{R} \times \bar{A}^c)) = \mathcal{H}^1(J_1^d \cap (\mathbb{R} \times \bar{A}^c))$$

where $\bar{A}^c = \mathbb{R} \setminus \bar{A}$. We also notice that $\mathcal{H}^1(J_1^d \cap (\mathbb{R} \times \partial A)) = 0$, so that in fact

$$\mathcal{H}^1(J_1^v \cap (\mathbb{R} \times A^c)) \geq \mathcal{H}^1(J_1^d \cap (\mathbb{R} \times A^c)).$$

For \mathcal{L}^1 -almost every $t \in A$, function $w : s \mapsto v(s, t)$ is differentiable on $]a, b[$ with derivative ± 1 , where a and b are such that $]a, t), (b, t)[= \Omega \cap (\mathbb{R} \times \{t\})$. Moreover, $-d((s, t)) < w(s) < d((s, t))$ for some $s \in]a, b[$, so that $w' = \frac{\partial v}{\partial x_1}$ has at least two jumps in $]a, b[$. As a consequence, applying Lemma 3.9, we get

$$\begin{aligned} \mathcal{H}^1(J_1^v \cap (\mathbb{R} \times A)) &\geq \int_{t \in A} \mathcal{H}^0(J_1^v \cap (\mathbb{R} \times \{t\})) dt \\ &\geq \int_{t \in A} 2 dt = 2 \mathcal{L}^1(A). \end{aligned}$$

On the other hand, the geometry of J_1^d implies that

$$\mathcal{H}^1(J_1^d \cap (\mathbb{R} \times A)) \leq \sqrt{2} \mathcal{L}^1(A).$$

As a consequence we get $\mathcal{H}^1(J_1^v) > \mathcal{H}^1(J_1^d)$. The inequality $\mathcal{H}^1(J_2^v) > \mathcal{H}^1(J_2^d)$ follows in the same way. \square

We observe that $d_{l^1}(\cdot, \partial\Omega)$ is not in general an optimal solution of (P_1) , as the following two examples show.

Example 3.10. Consider the subsets $\Omega_+ := \{x \in \mathbb{R}^2 : \|x\|_{l^1} \leq 3\}$ and $\Omega_- := \{x \in \mathbb{R}^2 : \|x - (2, 2)\|_{l^1} \leq 1\}$ and set $\Omega := \text{int}(\Omega_+ \cup \Omega_-)$. Define u as

$$u(x) = d_{l^1}(x, \partial\Omega_+) \chi_{\Omega_+}(x) - d_{l^1}(x, \partial\Omega_-) \chi_{\Omega_-}(x).$$

Then $\mathcal{F}(u) = 16$, while $\mathcal{F}(d_{l^1}(\cdot, \partial\Omega)) = 16 + 2\sqrt{2}$.

The fact that the function $d_{l^1}(\cdot, \partial\Omega)$ is not a solution of problem (P_1) is not a question of sign, as one could think from the previous example (notice that u is not positive on Ω). Indeed in the next example we exhibit a non-negative function $v \in \mathcal{E}_{pw}(\Omega)$ such that $\mathcal{F}(d_{l^1}(\cdot, \partial\Omega)) > \mathcal{F}(v)$.

Example 3.11. Define the four following squares $C_1 := \{x \in \mathbb{R}^2 : \|x\|_{l^1} \leq 12\}$, $C_2 := \{x \in \mathbb{R}^2 : \|x - (10, 10)\|_{l^1} \leq 8\}$, $C_3 := \{x \in \mathbb{R}^2 : \|x - (2, 11)\|_{l^1} \leq 1\}$, $C_4 := \{x \in \mathbb{R}^2 : \|x - (11, 2)\|_{l^1} \leq 1\}$. Let $\Omega := \text{int}(C_1 \cup C_2 \cup C_3 \cup C_4)$. It is easy to see that $\mathcal{F}(d_{l^1}(\cdot, \partial\Omega)) = 120 + 16\sqrt{2}$. Now, let us define

$$u(x) = d_{l^1}(x, \partial(C_1 \cup C_2))\chi_{C_1 \cup C_2}(x) + d_{l^1}(x, \partial C_3)\chi_{C_3}(x) + d_{l^1}(x, \partial C_4)\chi_{C_4}(x).$$

Then u is non-negative, u is a solution of (1.1) and $\mathcal{F}(u) = 88 + 24\sqrt{2}$, and thus $d_{l^1}(\cdot, \partial\Omega)$ is not optimal for (P_1) among the non-negative functions of $\mathcal{E}_{pw}(\Omega)$.

4. THE GENERAL CASE

We now turn to the general case where Ω is any bounded open connected subset of \mathbb{R}^N with Lipschitz boundary. In the previous section we saw that the functional \mathcal{F} attains its minimum in $\mathcal{E}_{pw}(\Omega)$ if the set Ω satisfies assumption (H) . In the general case we can't expect \mathcal{F} to be finite in $\mathfrak{S}(\Omega)$, as the following example shows.

Example 4.1. Let $\Omega = (0, 1)^2$: we claim that $\mathcal{F}(v) = +\infty$ for every $v \in \mathfrak{S}(\Omega)$. Indeed, let us consider, for $t \in (0, 1/2)$ fixed, the points (s, t) in Ω with $s \in (0, 1)$. Then for almost every $t \in (0, 1/2)$ the function $w_t : s \mapsto v(s, t)$ has the following properties: $w_t' = \frac{\partial v}{\partial x_1}(\cdot, t)$, $|w_t'| = 1$ a.e., $w_t(0) = w_t(1) = 0$ and $|w_t| \leq t$, due to Lemma 3.4. This implies that w_t' has at least $\lfloor \frac{1}{2t} \rfloor$ jumps, if $\lfloor x \rfloor$ denote the integer part of a real number x . Therefore, applying Lemma 3.9, we have

$$\mathcal{H}^1(J_1^v) \geq \int_0^{1/2} \mathcal{H}^0(J_1^v \cap ((0, 1) \times \{t\})) dt \geq \int_0^{1/2} \left\lfloor \frac{1}{2t} \right\rfloor dt = +\infty,$$

and so $\mathcal{F}(v) = \sum_{i=1}^2 \mathcal{H}^1(J_i^v) = +\infty$.

In view of the last example, a possible way to generalize the analysis of the previous section is to measure the jump set of Du for $u \in \mathfrak{S}(\Omega)$ through a "weight" function $h \in C_0(\Omega)$ which penalizes the jumps of Du in the interior of Ω .

We propose the following construction for the function h . Let $(\Omega_n)_{n \geq 1}$ be a sequence of open subsets of Ω satisfying hypothesis (H) and approximating Ω in the following way:

$$\forall n \geq 1 \quad \frac{1}{n+1/2} \leq d(\Omega_n, \partial\Omega) \leq \frac{1}{n}.$$

We set $\Omega_0 = \emptyset$. Notice that the sequence $(\Omega_n)_{n \geq 1}$ is increasing, and that for any $n \geq 1$ the open set $\omega_n := \Omega_n \setminus \overline{\Omega_{n-1}}$ satisfies hypothesis (H) . Then for any $n \geq 1$, consider the problem

$$(P_1^n) \quad \inf \left\{ \sum_{i=1}^N \mathcal{H}^{N-1}(J_i^v) : v \in \mathcal{E}_{pw}(\omega_n) \right\}.$$

It follows from Theorem 3.1 that $\delta_n := \inf(P_1^n) < +\infty$ for any fixed $n \geq 1$. We now define $h \in C_0(\Omega,]0, +\infty[)$ by

$$h(x) := \begin{cases} 1/[4(\alpha_1 + \alpha_2)] & \text{if } x \in \Omega_1, \\ \frac{1}{d_{l^1}(x, \partial\Omega_n) + d_{l^1}(x, \partial\Omega_{n+1})} \left[\frac{d_{l^1}(x, \partial\Omega_{n+1})}{(n+1)^2(\alpha_n + \alpha_{n+1})} + \frac{d_{l^1}(x, \partial\Omega_n)}{(n+2)^2(\alpha_{n+1} + \alpha_{n+2})} \right] & \text{if } x \in \omega_{n+1}, n \geq 1, \end{cases}$$

where $\alpha_n := \max\{1, \delta_n + \mathcal{H}^{N-1}(\partial\Omega_n)\}$ for any $n \geq 1$.

Define

$$\mathcal{E}(\Omega) := \{v \in \mathfrak{S}(\Omega) : Dv \in SBV_{loc}(\Omega)^N\}.$$

We will show the following result:

Theorem 4.2. *Let Ω be an open bounded connected subset of \mathbb{R}^N with Lipschitz boundary. Let h defined as above. Then the variational problem*

$$(P_h) \quad \inf \left\{ \mathcal{F}_h(v) := \sum_{i=1}^N \int_{J_i^v} h(x) d\mathcal{H}^{N-1}(x) : v \in \mathcal{E}(\Omega) \right\}.$$

has finite value and has an optimal solution.

Remark 4.3. As for problem (P_1) , (P_h) has at least two optimal solutions: if v is a solution, then $-v$ is a solution too.

Remark 4.4. If Ω satisfies hypothesis (H) , one can choose $h \equiv 1$ as Theorem 3.1 shows.

Proof. We divide the proof into two steps: in the first one we show that $\inf(P_h)$ is finite and in the second one we prove the existence of an optimal solution.

Step 1: We show that there exists a function $u \in \mathcal{E}(\Omega)$ such that $\mathcal{F}_h(u)$ is finite. For any $n \geq 1$ let ω_n be the sets considered in the definition of h . Let u_n be an optimal solution of the associated problem (P_1^n) , extended by 0 on $\mathbb{R}^N \setminus \omega_n$ (such a solution exists due to Theorem 3.1). We then define $u \in \mathcal{E}(\Omega)$ by $u := \sum_{n=1}^{+\infty} u_n(x)$, and claim that $\mathcal{F}_h(u) < +\infty$. Indeed, using that $h \leq \frac{2}{\alpha_n n^2}$ on $\bar{\omega}_n$ for any $n \geq 1$, one can estimate, for every i

$$\begin{aligned} \int_{J_i^u} h(x) d\mathcal{H}^{N-1}(x) &= \sum_{n=1}^{+\infty} \int_{J_i^u \cap (\omega_n \cup \partial\Omega_n)} h(x) d\mathcal{H}^{N-1}(x) \\ &= \sum_{n=1}^{+\infty} \left[\int_{J_i^{u_n} \cap \omega_n} h(x) d\mathcal{H}^{N-1}(x) + \int_{J_i^u \cap \partial\Omega_n} h(x) d\mathcal{H}^{N-1}(x) \right] \\ &\leq \sum_{n=1}^{+\infty} \frac{2}{n^2} \left[\int_{J_i^{u_n} \cap \omega_n} \frac{1}{\delta_n} d\mathcal{H}^{N-1}(x) + \int_{J_i^u \cap \partial\Omega_n} \frac{1}{\mathcal{H}^{N-1}(\partial\Omega_n)} d\mathcal{H}^{N-1}(x) \right] \\ &\leq \sum_{n=1}^{+\infty} \frac{4}{n^2} < +\infty. \end{aligned}$$

As a consequence $\inf(P_h)$ has finite value.

Step 2: We show that there exists an optimal solution of problem (P_h) applying the direct methods of the Calculus of Variations. Let $(v^n)_n \subset \mathcal{E}(\Omega)$ be a minimizing sequence for (P_h) . As in the proof of Theorem 3.1 we can assume that, up to a subsequence, $v^n \rightarrow v^\infty$ in $C_0(\Omega)$ and weakly* in $W^{1,\infty}(\Omega)$ for some $v^\infty \in W_0^{1,\infty}(\Omega)$. It remains to prove that v^∞ is an optimal solution of (P_h) .

Let us prove that $v^\infty \in \mathcal{E}(\Omega)$. Since $(v^n)_n$ is a minimizing sequence we can assume that there exists $C > 0$ such that $\mathcal{F}_h(v^n) \leq C$ for every $n \in \mathbb{N}$. Fix an open subset ω of Ω with Lipschitz boundary and such that $\omega \subset \bar{\omega} \subset \Omega$, and let $\alpha > 0$ be such that $0 < \alpha \leq h \leq 1$ in $\bar{\omega}$. We can then write

$$C \geq \mathcal{F}_h(v^n) \geq \alpha \sum_{i=1}^N \int_{J_i^{v^n} \cap \omega} d\mathcal{H}^{N-1}(x), \quad \forall n \in \mathbb{N}.$$

Let us fix $i \in \{1, \dots, N\}$. Since $\frac{\partial v^n}{\partial x_i} \in SBV(\omega)$ and takes only the two values ± 1 for any n , we can then write

$$\left\| \frac{\partial v^n}{\partial x_i} \right\|_{BV(\omega)} = \mathcal{L}^N(\omega) + 2\mathcal{H}^{N-1}(J_i^{v^n} \cap \omega) \leq \mathcal{L}^N(\omega) + 2C.$$

We now apply Theorem 6.2 to the sequence $(\frac{\partial v^n}{\partial x_i})_n$: condition *i*) is the inequality given above; condition *ii*) is straightforward since $\nabla \frac{\partial v^n}{\partial x_i} = 0$ a.e. in ω ; for condition *iii*) it is sufficient to choose $f \equiv 1$, for example, because in this way

$$\forall n \in \mathbb{N}, \quad \int_{J_i^{v^n} \cap \omega} f \left(\left[\frac{\partial v^n}{\partial x_i} \right] \right) d\mathcal{H}^{N-1}(x) = \mathcal{H}^{N-1}(J_i^{v^n} \cap \omega) \leq C.$$

As a consequence $\frac{\partial v^n}{\partial x_i} \rightharpoonup g_i$ weakly* in $BV(\omega)$ for some function $g_i \in SBV(\omega)$. Since we already know that $\frac{\partial v^n}{\partial x_i} \rightharpoonup \frac{\partial v^\infty}{\partial x_i}$ weakly* in $L^\infty(\Omega)$, we infer that $\frac{\partial v^\infty}{\partial x_i} \in SBV(\omega)$. Moreover $\frac{\partial v^n}{\partial x_i} \rightarrow \frac{\partial v^\infty}{\partial x_i}$ in $L^1(\omega)$ so that $|\frac{\partial v^\infty}{\partial x_i}| = 1$ a.e. in ω . This being true for any index i and any such open subset ω of Ω , we infer that v^∞ belongs to $\mathcal{E}(\Omega)$.

To prove that v^∞ is a minimizer of \mathcal{F}_h , let us show that

$$\liminf_{n \rightarrow \infty} \int_{J_i^{v^n} \cap \omega} h(x) d\mathcal{H}^{N-1}(x) \geq \int_{J_i^{v^\infty} \cap \omega} h(x) d\mathcal{H}^{N-1}(x) \quad i = 1, \dots, N \quad (4.1)$$

for any ω as above. To this aim we observe that for every $n \in \mathbb{N} \cup \{\infty\}$ one has

$$\int_{J_i^{v^n} \cap \omega} h(x) d\mathcal{H}^{N-1}(x) = \frac{1}{2} \int_{\omega} h(x) d \left| D \frac{\partial v^n}{\partial x_i} \right| (x).$$

Therefore (4.1) follows from Theorem 6.4. This is sufficient to conclude the proof, since

$$\liminf_{n \rightarrow \infty} \int_{J_i^{v^n} \cap \omega} h(x) d\mathcal{H}^{N-1}(x) \geq \sup_{\omega \subset \Omega} \int_{J_i^{v^\infty} \cap \omega} h(x) d\mathcal{H}^{N-1}(x) = \int_{J_i^{v^\infty} \cap \omega} h(x) d\mathcal{H}^{N-1}(x)$$

where the supremum is taken over the open subsets $\omega \subset \bar{\omega} \subset \Omega$ with Lipschitz boundary. We then infer from the last inequality that $\liminf_{n \rightarrow \infty} \mathcal{F}_h(v^n) \geq \mathcal{F}_h(v^\infty)$, that is v^∞ is a minimizer of \mathcal{F}_h . \square

5. OPEN PROBLEMS

We would like to present here some open problems related to our work: some questions about problems (P_1) and (P_h) and the question of the selection of a particular solution of the differential problem (1.1).

1. Suppose that Ω satisfies property (H) . We showed in section 3 that there exists a function $u \in C_{pw}^1(\Omega)$ solution of problem (P_1) . Can we say that the number of the connected components of Du is finite? This seems to be natural, as shown in Lemma 3.5.

Another natural question is: has problem (P_1) a finite number of solutions? Does it admit just two optimal solutions (a solution and its opposite)? This is true when Ω is a rectangle and $N = 2$ as shown in Proposition 3.8. With the same arguments as in the proof of this proposition, it is easy to show that the functions $\pm u$, where u is the solution defined in Example 3.10, are the only optimal solutions of problem (P_1) .

Another related question is to extend the result of Proposition 3.8 to higher dimension, that is to prove that if Ω a convex open subset of \mathbb{R}^N satisfying (H) then the distance functions $\pm d_{l^1}(\cdot, \partial\Omega)$ are the only solutions of problem (P_1) . We notice that under the above hypothesis on Ω , the function $d_{l^1}(\cdot, \partial\Omega)$ is a concave solution of (1.1): this implies that on a non-void interval $\mathbb{R} \times \{y\} \cap \Omega$ (where $y \in \mathbb{R}^{N-1}$) the partial derivative $\frac{\partial}{\partial x_1} d_{l^1}(\cdot, \partial\Omega)$ has only one jump. As a consequence the jump set of $\frac{\partial}{\partial x_1} d_{l^1}(\cdot, \partial\Omega)$ is also in some sense minimal since on each such interval it is reduced to one point, and so it has minimal cardinality.

2. In section 4 we studied the general case, that is when Ω is any subset of \mathbb{R}^N . A natural question is the study of the asymptotic behavior of a family $(u_h)_h$ of the optimal solutions of (P_h) when $h \rightarrow 1$ locally uniformly in Ω . The limit problem is then obviously (P_1) , which in general is ill-posed (since it has no feasible solution), but one expects u_h to converge uniformly to a function $u \in \mathcal{E}(\Omega)$ having some local minimization properties in the open subsets $\omega \subset \bar{\omega} \subset \Omega$.

3. In section 4 we saw that for a general open set Ω , $\mathcal{F}(v)$ may be infinite for any $v \in \mathfrak{S}(\Omega)$. This means that for some index i one has $\mathcal{H}^{N-1}(J_i^v) = \infty$. On the other hand we know that $\mathcal{H}^N(J_i^v) = 0$, so a natural question is to identify the Hausdorff dimension $s \in [N-1, N]$ for which $\mathcal{H}^t(J_i^v)$ is finite if $t < s$, infinite if $t > s$. Then one may try to minimize this dimension s over $\mathfrak{S}(\Omega)$, and then proceed as in section 3 with the optimal s in place of $N-1$.

4. In this article we proposed to minimize the measure of the discontinuity sets of the gradients of the elements of $\mathfrak{S}(\Omega)$ through the functionals \mathcal{F} and \mathcal{F}_h . Now, let $\{G_i\}_{i \in I}$ be the connected components of the $Du^{-1}(E)$ for $u \in \mathfrak{S}(\Omega)$, where we recall that $E = \{x : \forall i, |x_i| = \frac{1}{\sqrt{N}}\}$. One could instead consider functionals of the form

$$\sum_{i \in I} g(\mathcal{H}^{N-1}(\partial G_i)) \quad \text{with} \quad g(t) = \begin{cases} t^\beta, & t \leq 1 \\ t, & t \geq 1 \end{cases}$$

where $\beta \in]0, 1[$. The advantage of this last type of functional with respect to our choice is that we can penalize the number of components G_i thanks to g .

5. One of the aim of this work was to select a particular solution of (1.1). The idea developed in the previous section is first to restrict ourselves to those solutions of (1.1) that minimize problem (P_h) . Now how can we select a particular solution of (P_h) (up to its sign)? Does problem (P_h) admit more than two solutions? Notice that in any case the selection of these two solutions of problem (1.1) depends on the choice of h , so that it would be even more interesting to select a solution obtained by letting h go to 1 (see 2. above).

In the case where problem (P_h) admits more than two solutions one can think of the following way to select a particular solution. Let \mathcal{S}_0 be the set of solutions of problem (P_h) , and let $(f_n)_{n \geq 1}$ be an orthogonal basis of $L^2(\Omega)$ starting from $f_1 = 1$. With the same arguments as in the proof of Theorem 4.2, it is not difficult to show that problem

$$(B_1) \quad \sup \left\{ \int_{\Omega} u f_1, u \in \mathcal{S}_0 \right\}$$

admits a solution. Let \mathcal{S}_1 be the set of optimal solutions of problem (B_1) . If \mathcal{S}_1 is composed by more than one function, we consider

$$(B_2) \quad \sup \left\{ \int_{\Omega} u f_2, u \in \mathcal{S}_1 \right\}$$

and so on. If two functions $u, v \in \mathcal{S}_0$ satisfy

$$\int_{\Omega} u f_i = \int_{\Omega} v f_i \quad \forall i \geq 1$$

then one gets $u = v$. Therefore this method allows us to select a particular solution of (P_h) .

6. APPENDIX

We recall here some compactness and semicontinuity results about BV , SBV and piecewise constant functions; for the proofs, see respectively Theorem 3.23 in [2], Theorem 1.4 in [1], Theorem 4.25 and Theorem 2.38 in [2].

Theorem 6.1 (compactness for BV functions). *Let Ω be a bounded open subset of \mathbb{R}^N with Lipschitz boundary. Assume that $(u_n)_n$ is an uniformly bounded sequence in $BV(\Omega)$. Then there exist a subsequence $(u_{n_k})_k$ and a function $u \in BV(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly* in $BV(\Omega)$.*

Theorem 6.2 (compactness for SBV functions). *Let Ω be a bounded open subset of \mathbb{R}^N with Lipschitz boundary. Let $(u_n)_n$ be a sequence of functions in $SBV(\Omega)$ and assume that*

- i) the functions u_n are uniformly bounded in $BV(\Omega)$;*
- ii) the gradients ∇u_n are equi-integrable;*
- iii) there exists a function $f : [0, \infty) \rightarrow [0, \infty]$ such that $f(t)/t \rightarrow \infty$ as $t \rightarrow 0^+$ and*

$$\int_{J_{u_n}} f([u_n]) d\mathcal{H}^{N-1} \leq C < \infty \quad \forall n \in \mathbb{N}.$$

Then there exists a subsequence $(u_{n_k})_k$ and a function $u \in SBV(\Omega)$ such that $u_{n_k} \rightharpoonup u$ weakly in $BV(\Omega)$, with the Lebesgue and jump parts of the derivatives converging separately, i.e. $D^a u_{n_k} \rightharpoonup D^a u$ and $D^j u_{n_k} \rightharpoonup D^j u$ weakly* in $\mathcal{M}(\Omega, \mathbb{R}^N)$.*

Theorem 6.3 (compactness for piecewise constant functions). *Let Ω be a bounded open subset of \mathbb{R}^N with Lipschitz boundary. Let $(u_n)_n$ be a sequence of piecewise constant functions in Ω such that $\|u_n\|_{L^\infty(\Omega)} + \mathcal{H}^{N-1}(J_{u_n})$ is uniformly bounded. Then there exist a subsequence $(u_{n_k})_k$ and a piecewise constant function u such that $u_{n_k} \rightarrow u$ in measure.*

Theorem 6.4 (semicontinuity in BV). *Let Ω be a bounded open subset of \mathbb{R}^N . Let $(u_n)_n$ be a sequence of functions in $BV(\Omega)$ such that $u_n \rightharpoonup u$ weakly* in $BV(\Omega)$. Then*

$$\int_{\Omega} f(x) d|Du|(x) \leq \liminf_{n \rightarrow \infty} \int_{\Omega} f(x) d|Du_n|(x)$$

for any non-negative continuous function $f : \Omega \rightarrow [0, +\infty[$.

Acknowledgments. The authors wish to express their thanks to Grégoire Charlot, Bernard Dacorogna and Jérôme Droniou for some stimulating conversations.

REFERENCES

- [1] Giovanni Alberti and Carlo Mantegazza. A note on the theory of SBV functions. *Boll. Un. Mat. Ital. B (7)*, 11(2):375–382, 1997.
- [2] Luigi Ambrosio, Nicola Fusco, and Diego Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.
- [3] Giovanni Bellettini, Bernard Dacorogna, Giovanni Fusco, and Francesco Leonetti. Qualitative properties of lipschitz solutions of eikonal type systems. *Adv. Math. Sci. Appl.*, (to appear), 2006.
- [4] Ariela Briani and Andrea Davini. Monge solutions for discontinuous Hamiltonians. *ESAIM Control Optim. Calc. Var.*, 11(2):229–251 (electronic), 2005.
- [5] Pierre Cardaliaguet, Bernard Dacorogna, Wilfrid Gangbo, and Nicolas Georgy. Geometric restrictions for the existence of viscosity solutions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 16(2):189–220, 1999.
- [6] Arrigo Cellina. On minima of a functional of the gradient: sufficient conditions. *Nonlinear Anal.*, 20(4):343–347, 1993.
- [7] Bernard Dacorogna, Roland Glowinski, Yuri Kuznetsov, and Tsorng-Whay Pan. On a conjugate gradient/Newton/penalty method for the solution of obstacle problems. Application to the solution of an eikonal system with Dirichlet boundary conditions. In *Conjugate gradient algorithms and finite element methods*, Sci. Comput., pages 263–283. Springer, Berlin, 2004.
- [8] Bernard Dacorogna, Roland Glowinski, and Tsorng-Whay Pan. Numerical methods for the solution of a system of eikonal equations with Dirichlet boundary conditions. *C. R. Math. Acad. Sci. Paris*, 336(6):511–518, 2003.
- [9] Bernard Dacorogna and Paolo Marcellini. *Implicit partial differential equations*. Progress in Nonlinear Differential Equations and their Applications, 37. Birkhäuser Boston Inc., Boston, MA, 1999.
- [10] Bernard Dacorogna and Paolo Marcellini. Viscosity solutions, almost everywhere solutions and explicit formulas. *Trans. Amer. Math. Soc.*, 356(11):4643–4653 (electronic), 2004.
- [11] Lawrence C. Evans and Ronald F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.
- [12] Herbert Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
- [13] Giovanni Pisante. Sufficient conditions for the existence of viscosity solutions for nonconvex Hamiltonians. *SIAM J. Math. Anal.*, 36(1):186–203 (electronic), 2004.

LABORATOIRE D'ANALYSE NON LINÉAIRE APPLIQUÉE, U.F.R. DES SCIENCES ET TECHNIQUES,
UNIVERSITÉ DE TOULON ET DU VAR, AVENUE DE L'UNIVERSITÉ, BP 132, 83957 LA GARDE
CEDEX, FRANCE

INSTITUT DE MATHÉMATIQUES ET DE MODÉLISATION DE MONTPELLIER, UNIVERSITÉ MONT-
PELLIER II MONTPELLIER FRANCE