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## ON THE POWER LAW ASYMPTOTICS

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**Abstract:** In this paper we analyze a model of dielectric breakdown by adapting a technique which was recently introduced in [6].

**1. Introduction.** In [6] a general homogenization result was proved for variational functionals with constraints on the gradient. Here we restrict our attention to a model of dielectric breakdown introduced in [8]. The results of [6] do not apply here since boundary conditions are not of the same type: homogeneous Dirichlet condition in [6] versus periodic boundary condition in the present note. However the technique used in [6] is adapted and in particular the use of a monotone approximation for the proof of the  $\Gamma$ -lim inf estimate. Here we choose to use  $L^p$  (or power law) approximation instead of the Moreau-Yosida approximation. This is appropriate in view of the rescaling properties of  $L^p$  approximation. The idea of  $L^p$ -approximations is also contained in [8] although applied in a different way and with the aim of obtaining some optimal bound.

Let us describe the problem. The behaviour of a body subject to a given electric field  $\nabla u$  can be modelled in various ways. A common way to do it is (see [8] for more references) to associate a suitable convex set  $K(x)$  at each point of the body and to distinguish the behaviour of the body with respect to a given electric field  $\nabla u$  considering if the condition  $\nabla u(x) \in K(x)$  a.e. is satisfied or not. If  $\nabla u(x) \in K(x)$  a.e. is satisfied then the body behaves as an insulator, if  $\nabla u(x) \in K(x)$  a.e. is not satisfied then the dielectric breakdown occurs and the body starts to conduct. Therefore a pointwise constraint on the gradient play the main role. In [8] the authors propose to replace the pointwise constraint with a less degenerate supremal functional associated to the convex set  $K_{eff}$  of all average electric fields  $\xi$  for which there exists  $u$  such that  $\xi = \frac{1}{|\Omega|} \int \nabla u dx$  and  $\nabla u(x) \in K(x)$  almost everywhere. Then they obtain optimal bounds for  $K_{eff}$ . Here we would like to find  $K_{eff}$  using the  $\Gamma$ -convergence and homogenization theory. To avoid confusion (following the example of [8]) we will call  $K^{hom}$  the convex set that play the role of  $K_{eff}$  in the model. The body will be represented by the unitary  $n$ -cube

$I^n := [0, 1]^n$  and  $K^{hom}$  will be defined by

$$\xi \in K^{hom} \Leftrightarrow u \in W_{\#}^{1,\infty}(I^n) \text{ s.t. } \nabla u(x) + \xi \in K(x) \text{ a.e..}$$

Throughout the paper we will always follow the convention that a function in  $W^{1,\infty}$  is identified with its continuous representant. We denote by  $\chi_A$  the characteristic function of the set  $A$  (i.e.  $\chi_A(x) = 0$  if  $x \in A$ ,  $\chi_A(x) = \infty$  otherwise). We will prove the following homogenization result:

**Theorem 1.1** *The family of functionals  $(\chi_{\{\nabla u(x) \in K(\frac{x}{\varepsilon})\}})_{\varepsilon}$   $\Gamma$ -converges to  $\chi_{\{\nabla u(x) \in K^{hom}\}}$  on the space  $W_{\#}^{1,\infty}(I^n)$  equipped with the uniform convergence of continuous functions as  $\varepsilon \rightarrow 0$ .*

**Remark.** As the functionals we consider have sublevel sets which are (up to addition of constant functions) compact in  $C^0(I^n)$  endowed with the sup norm topology (this is a consequence of Ascoli-Arzelà theorem), the  $\Gamma$ -limit result also holds with the  $L^1$  norm.

To be short and simplify the notations we will denote  $\chi_{\{\nabla u(x) \in K(\frac{x}{\varepsilon})\}}$  by  $\chi^{\varepsilon}$  and

$\chi_{\{\nabla u(x) \in K^{hom}\}}$  by  $\chi^{hom}$ . The result we prove is already known (see [5] for example) but the proof we give here is simpler than the others in litterature. We split the proof in two parts: first we prove the  $\Gamma$  – lim inf inequality by approximating  $\chi^{\varepsilon}$  by standard energies  $F_p^{\varepsilon}$  with  $p$ -growth, then we show the  $\Gamma$  – lim sup inequality using a constructive method introduced in [6].

**2.  $\Gamma$  – liminf inequality.** In this section we prove the following theorem:

**Theorem 2.1.** *Let  $(u_{\varepsilon})_{\varepsilon}$  be a family in  $W_{\#}^{1,\infty}(I^n)$  which converges uniformly to  $u$  then  $\liminf_{\varepsilon \rightarrow 0} \chi^{\varepsilon}(u_{\varepsilon}) \geq \chi^{hom}(u)$ .*

The above theorem states that  $\Gamma$  – lim inf  $\chi^{\varepsilon} \geq \chi^{hom}$ . Our proof of theorem 2.1 is based on the  $L^p$  approximation: for any positive  $\varepsilon$  we introduce the approximating energies  $F_p^{\varepsilon}$  defined on  $C^0(I^n)$  by:

$$F_p^{\varepsilon}(u) := \begin{cases} \frac{1}{p} \int_{\Omega} (\phi(\frac{x}{\varepsilon}, Du(x))^p - 1) dx & \text{if } u \in W_{\#}^{1,p}(I^n), \\ +\infty & \text{otherwise,} \end{cases}$$

where  $n < p$  and  $\phi(x, \cdot)$  denotes the gauge function of  $K(x)$  (i.e.  $\phi(x, \cdot)$  is the only non-negative homogeneous convex function such that  $\phi(x, \xi) \leq 1 \Leftrightarrow \xi \in K(x)$ ).

The  $L^p$  approximation scheme is monotone thanks to the following lemma:

**Lemma 2.2.** *For all numbers  $0 \leq a$  the function  $p \mapsto \frac{1}{p}a^p - \frac{1}{p}$  is non-decreasing in  $p$  and  $\lim_{p \rightarrow \infty} \frac{1}{p}a^p - \frac{1}{p}$  is equal to 0 if  $a \leq 1$  and to  $+\infty$  if  $1 < a$ .*

**Proof:** The lemma is straightforward in the case  $a = 0$ . When  $a > 0$ , then the convexity of the exponential function yields

$$\forall p > q > 1 \quad \exp\left(\frac{p}{q} \ln(a^q) + \left(1 - \frac{p}{q}\right) \ln(1)\right) \leq \frac{p}{q} a^q + \left(1 - \frac{p}{q}\right)$$

from which the lemma follows easily.  $\square$

As a direct consequence we get the following

**Proposition 2.3.** *For any positive  $\varepsilon$  the family  $(F_p^\varepsilon)_p$   $\Gamma$ -converges in  $C^0(I^n)$  to  $\chi^\varepsilon$  non-decreasingly when  $p \rightarrow \infty$ .*

The previous proposition is in some sense the non homogeneous, anisotropic version of the well known theorem on the limit of the  $p$ -laplacian for  $p \rightarrow \infty$ .

For fixed  $p$ , the functionals  $F_p^\varepsilon$  satisfy the  $p$ -growth conditions

$$\frac{1}{p} \int_{I^n} \left( \left( \frac{|Du|(x)}{R} \right)^p - 1 \right) dx \leq F_p^\varepsilon(u) \leq \frac{1}{p} \int_{I^n} \left( \left( \frac{|Du|(x)}{r} \right)^p - 1 \right) dx.$$

Then the following classical homogenization result holds (see chapter 14 of [4])

$$(\Gamma - \lim_{\varepsilon \rightarrow 0} F_p^\varepsilon)(u) = \int_{I^n} \phi_p(Du(x)) dx,$$

where the integrand  $\phi_p$  is given by the formula:

$$\phi_p(\xi) = \inf_{v \in W_{\#}^{1,p}} \frac{1}{p} \int_{I^n} (\phi(x, \xi + Dv)^p - 1) dx.$$

**Proof (of theorem 2.1).** Let  $u_\varepsilon \rightarrow u$  as in the statement. Using proposition 2.2 and what precedes we get

$$\forall p \quad \liminf_{\varepsilon \rightarrow 0} \chi^\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon \rightarrow 0} F_p^\varepsilon(u_\varepsilon) \geq F_p(u).$$

Then taking the limit in  $p$  and applying lemma 2.2 yield

$$\liminf_{\varepsilon \rightarrow 0} \chi^\varepsilon(u_\varepsilon) \geq \lim_{p \rightarrow \infty} F_p(u) = \chi^{hom}(u).$$

$\square$

**3.  $\Gamma$ -limsup inequality.** In this section we prove the following result:

**Theorem 3.1.** *Let  $u \in W_{\#}^{1,\infty}(I^n)$  be such that  $\chi^{hom}(u) = 0$ , then there exists a family  $(u_\varepsilon)_\varepsilon \subset W_{\#}^{1,\infty}(I^n)$  converging to  $u$  uniformly and such that  $\chi^\varepsilon(u_\varepsilon) = 0$ .*

It follows from the above result and theorem 2.1 that  $\Gamma - \limsup_\varepsilon \chi^\varepsilon \leq \chi^{hom}$ .

**Proof. Step 1:** Let  $u \in W_{\#}^{1,\infty}(I^n)$  be piecewise affine and such that  $\chi^{hom}(u) = 0$ . We claim that there exists a family  $(u_\varepsilon)_\varepsilon$  in  $W_{\#}^{1,\infty}(I^n)$  converging uniformly to  $u$  and such that  $\chi^\varepsilon(u_\varepsilon) = 0$  for any  $\varepsilon > 0$ .

Since  $u$  is piecewise affine, there exists a family  $(\omega_i)_{1 \leq i \leq k}$  of disjoint open sets such that  $\overline{I^n} = \cup_{1 \leq i \leq n} \overline{\omega_i}$  and  $u(x) = \alpha_i + \xi_i \cdot x$  for any  $x$  in  $\overline{\omega_i}$ . For

any  $z$  in  $\{-1, 0, 1\}^n$ , we define  $u^{i,z}$  on  $\mathbb{R}^n$  by  $u^{i,z} : x \mapsto \alpha_i + \xi_i \cdot (x - z)$ . By periodicity of  $u$ , one has  $u(x) = u^{i,z}(x)$  for any  $x$  in  $z + I^n$ . Then  $u$  can be written as a finite combination of *min* and *max* of the functions  $u^{i,z}$  on the set  $[-1, 2]^n$ : we shall write formally this combination  $u = c(u^{i,z} : z \in \{-1, 0, 1\}^n, 1 \leq i \leq k)$ .

For any  $i$  in  $\{1, \dots, k\}$ ,  $\xi_i$  belongs to  $K^{hom}$  so that there exists a function  $w_i$  in  $W_{\#}^{1,\infty}(I^n)$  such that  $\xi_i + \nabla w_i(x)$  belongs to  $K(x)$  a.e. in  $I^n$ . For any  $\varepsilon > 0$  and  $z \in \{-1, 0, 1\}^n$ , we define the function  $u_{\varepsilon}^{i,z}$  on  $\mathbb{R}^n$  by  $u_{\varepsilon}^{i,z} : x \mapsto u^{i,z}(x) + \varepsilon w_i(\frac{x}{\varepsilon})$ . We then set  $\tilde{u}_{\varepsilon} := c(u_{\varepsilon}^{i,z} : z \in \{-1, 0, 1\}^n, 1 \leq i \leq k)$  for any  $\varepsilon > 0$ , i.e.  $\tilde{u}_{\varepsilon}$  is defined through the same formal combination of *min* and *max* as  $u$ . We remark that for two points  $x, y$  in  $I^n$  for which there exists  $z \in \{0, 1\}^n$  such that  $x = y + z$ , the active functions in the combination  $c(u^{i,z} : z \in \{-1, 0, 1\}^n, 1 \leq i \leq k)$  that define  $u(x)$  and  $u(y)$  are the same up to the translation of vector  $z$ . The same holds true for  $\tilde{u}_{\varepsilon}$ , so that the restriction  $u_{\varepsilon}$  of  $\tilde{u}_{\varepsilon}$  to  $I^n$  is a function of  $W_{\#}^{1,\infty}(I^n)$ . We now notice that of course one has  $\chi^{\varepsilon}(u_{\varepsilon}) = 0$  for any  $\varepsilon > 0$ , and it is clear that the family  $(u_{\varepsilon})_{\varepsilon}$  converges uniformly to  $u$  on  $I^n$ .

Step 2: Let  $u \in C_{\#}^{\infty}(I^n)$  be such that  $\chi^{hom}(u) = 0$ . We claim that there exists a sequence  $(u_n)_n$  of functions in  $W_{\#}^{1,\infty}(I^n)$  which are piecewise affine, satisfy  $\chi^{hom}(u_n) = 0$  for any  $n \in \mathbb{N}$  and converge uniformly to  $u$  on  $I^n$ .

Since  $u$  is smooth,  $K^{hom}$  is closed and  $\chi^{hom}(u) = 0$  then  $\nabla u(x) \in K^{hom}$  for any  $x$  in  $I^n$ . Moreover thanks to the smoothness of  $u$  there exists a sequence of functions  $(u_n)_n$  which are periodic, piecewise affine and such that  $u_n \rightarrow u$  strong in  $W^{1,\infty}$ . As  $K$  is a convex set of nonempty interior it is possible to choose  $u_n$  so that it satisfies  $\chi^{hom}(u_n) = 0$ .

Step 3: Let now  $u$  in  $W_{\#}^{1,\infty}(I^n)$  be such that  $\chi^{hom}(u) = 0$ . We claim that there exists a family  $(u_{\varepsilon})_{\varepsilon}$  in  $W_{\#}^{1,\infty}(I^n)$  converging uniformly to  $u$  and such that  $\chi^{\varepsilon}(u_{\varepsilon}) = 0$  for any  $\varepsilon > 0$ . To prove this we fix a positive real number  $\eta$  and show that there exists a function  $u_{\eta}$  in  $W_{\#}^{1,\infty}(I^n)$  which is piecewise affine and satisfies  $\chi^{hom}(u_{\eta}) = 0$  as well as  $\|u - u_{\eta}\|_{\infty} \leq \eta$ . Since this is true for any  $\eta > 0$ , step 1 and a standard diagonal argument prove the claim.

Now fix  $\eta > 0$ . Let  $\gamma \in C_c^{\infty}(\mathbb{R}^n, \mathbb{R}_+)$  be a standard mollifier (i.e. with  $\int_{\mathbb{R}^n} \gamma(x) dx = 1$ ) and  $(\gamma_{\delta})_{\delta > 0}$  the associated approximation of unity (i.e.  $\gamma_{\delta}(\cdot) := \delta^n \gamma(\cdot/\delta)$ ). For any  $\delta > 0$ , we set  $v_{\delta} = \gamma_{\delta} * u$  (where  $u$  is defined on  $\mathbb{R}^n$  by periodicity), then the function  $v_{\delta}$  belongs to  $W_{\#}^{1,\infty}(I^n)$  and we conclude from the identity

$$\forall x \in \mathbb{R}^N \quad Dv_{\delta}(x) = \int_{\mathbb{R}^N} \gamma_{\delta}(x - y) Du(y) dy$$

that  $Dv_{\delta}(x)$  belongs to  $K^{hom}$  for any  $x$  in  $I^n$ . As a consequence, for any positive  $\delta$  the function  $v_{\delta}$  belongs to  $C^{\infty}(I^n)$  and is such that  $\chi^{hom}(v_{\delta}) = 0$ . Since the family  $(v_{\delta})_{\delta}$  converges uniformly to  $u$  on  $I^n$ , the existence of the

desired function  $u_\eta$  follows easily from step 2 applied to  $v_\delta$  for  $\delta > 0$  small enough.  $\square$

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