Study of some variants of the Monge-Kantorovich optimal transport problem

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In optimal mass transport theory, many problems can be written in the Monge-Kantorovich form

$$\inf\{\int_{X\times Y} c(x,y)\,d\gamma \;:\; \gamma\in\Pi(\mu,\nu)\}\ , \tag{1}$$

where μ,ν are given probability measures on X,Y and $c: X\times Y\to[0,+\infty[$ is a cost function. Here the competitors are probability measures γ on $X\times Y$ with marginals μ and ν respectively (transport plans). Let us recall that if an optimal transport plan $\gamma\in\Pi(\mu,\nu)$ is carried by the graph of a map $T: X\to Y$ i.e. if

$$<\gamma,\varphi(x,y)>=\int_X \varphi(x,Tx)\,d\mu \quad , \quad T^\sharp\mu=\nu\ ,$$

then T solves the original Monge problem : $\inf\{\int_X c(x,Tx)\,d\mu \;:\; T^\sharp\mu=\nu\}$.

Here we are interested in a different case. Indeed in some applications to economy or in probability theory, it can be interesting to favour optimal plans which are non associated to a single valued transport map $T(x)$. The idea is then to consider, instead of $T(x)$, the family of conditional probabilities γ^x such that

$$<\gamma,\varphi(x,y)>=\int_X (\int_X \varphi(x,y)d\gamma^x(y))\,d\mu\ ,$$

and to incorporate in problem (1) an additional cost over γ^x as follows

$$\inf\left\{\int_{X\times X} c(x,y)\,d\gamma+\int_X G(x,\gamma^x)\,d\mu \;:\; \gamma\in\Pi(\mu,\nu)\right\}\ , \tag{2}$$

being $G:(x,p)\in X\times\mathcal{P}(X)\to[0,+\infty]$ a given non linear function.