3D-2D Analysis for the optimal elastic compliance problem

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The framework of linear elasticity

A displacement field $u$ on $\Omega$
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- The (linearized) strain tensor field $e(u) = (\nabla u + \nabla u^T)/2$
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- The (linearized) strain tensor field $e(u) = (\nabla u + \nabla u^T)/2$
- A quadratic coercive energy density $j(e(u))$
  (e.g. $j(e(u)) = \lambda(\text{tr}(e(u))^2 + \mu/2 \|e(u)\|^2)$)
The framework of linear elasticity

- A force distribution $F$ (volume or surface density, inside $\Omega$ or on its boundary)
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- $u \in H^1(\Omega)$, $F \in H^{-1}(\Omega)$.
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The equilibrium problem

\[
div \frac{\partial j}{\partial e}(e(u)) + F = 0 \text{ on } \Omega
\]
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$$-C = \inf_u \{ \int_\Omega j(e(u)) - \langle F, u \rangle \}$$

$$C = \sup_u \{ \langle F, u \rangle - \int_\Omega j(e(u)) \}$$
The framework of linear elasticity

$C$ is the “compliance” = the stored energy at equilibrium

- $C$ is positive, depends on $\Omega$, $j$ and $F$. 
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- \( C \) is positive, depends on \( \Omega, j \) and \( F \).
- \( C \) is finite if \( F \) is balanced (\( < F, u > = 0 \) whenever \( u \) is rigid) (\( \int F = 0, \int x \wedge F = 0 \))
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**Dual formulation**

$$C = \inf \left\{ \int_{\Sigma} j^*(\Sigma) ; \text{div} (\Sigma) + F = 0 \right\}$$

$$j^*(A) := \sup_B \{ A \cdot B - j(B) \}$$
\[ Q_\delta = \mathcal{D} \times \left[ -\frac{\delta}{2}, \frac{\delta}{2} \right], \quad \delta \to 0 \]
**3D-2D asymptotic analysis**

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**Question:**

\[ \lim_{\delta \to 0} \inf_u \left\{ \int_{Q_\delta} j(e(u)) - \langle F^\delta, u \rangle \right\} ? \]
$Q_\delta = D \times [\frac{-\delta}{2}, \frac{\delta}{2}], \quad \delta \to 0$

Question:

$$\lim_{\delta \to 0} \inf_u \left\{ \int_{Q_\delta} j(e(u)) - < F^\delta, u > \right\}$$

Rescaling:

$$y_\alpha = x_\alpha \ (\alpha = 1, 2), \quad y_3 = \frac{x_3}{\delta}$$
3D-2D asymptotic analysis

\[ u_\alpha(x_\alpha, x_3) = v_\alpha(x_a, \frac{x_3}{\delta}), \quad u_3(x_\alpha, x_3) = \frac{1}{\delta} v_3(x_a, \frac{x_3}{\delta}). \]
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\[ e(u)(x_\alpha, x_3) = e^\delta(v) := \begin{pmatrix} e_{\alpha,\beta}(v) & \frac{1}{\delta} e_{\alpha,3}(v) \\ \frac{1}{\delta} e_{\alpha,3}(v) & \frac{1}{\delta^2} e_{3,3}(v) \end{pmatrix} \]
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\[ e(u)(x_\alpha, x_3) = e^\delta(v) := \begin{pmatrix} e_{\alpha, \beta}(v) & \frac{1}{\delta} e_{\alpha, 3}(v) \\ \frac{1}{\delta} e_{\alpha, 3}(v) & \frac{1}{\delta^2} e_{3, 3}(v) \end{pmatrix} \]

\[ F^\delta_\alpha(x_\alpha, x_3) = \frac{1}{\sqrt{\delta}} F_\alpha(x_a, \frac{x_3}{\delta}), \quad F^\delta_3(x_\alpha, x_3) = \sqrt{\delta} F_3(x_a, \frac{x_3}{\delta}), \]
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Rescaled problem:

\[ \inf_v \left\{ \int_{Q_1} j(e^\delta(v)) - \langle F, v \rangle \right\} \]
Definition : $KL := \{v \in H^1(Q_1)/e_{\alpha,3}(v) = 0, e_{3,3}(v) = 0\}$

Characterization : $v$ is in $KL$ if and only if there exists $w$ such that $w_\alpha \in H^1(D)$, $w_3 \in H^2(D)$ and

$$v_3(y_\alpha, y_3) = w_3(y_\alpha), \quad v_\alpha(y_\alpha, y_3) = w_\alpha(y_\alpha) - \frac{\partial w_3}{\partial y_\alpha} y_3$$
3D-2D asymptotic analysis

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\]

- replace \( v \) by a KL-displacement in the infimum pb.
- explicit the energy in term of \( w \)
3D-2D asymptotic analysis

- gives the 2D plate problem:

\[
\inf_w \left\{ \int_D \mathcal{W}(e_{\alpha,\beta}(w_\alpha), \nabla^2 w_3) - \overline{F}, w > \right\}
\]

where

\[
\overline{F}_\alpha(y_\alpha) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F_\alpha(y_\alpha, y_3) \, dy_3
\]

\[
\overline{F}_3(y_\alpha) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F_3 + y_3 \frac{\partial F_1}{\partial y_1} + x_3 \frac{\partial F_2}{\partial y_2} \, dy_3
\]

and \( \mathcal{W} \) is explicit in terms of \( j \).
3D-2D asymptotic analysis

-> many extensions (non isotropic, non homogeneous, varying thickness, non linear cases)

-> but no simple extension with high contrast composites (or with voids)
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-> but no simple extension with high contrast composites (or with voids)
Shape optimization

In a given design region $Q$, a given amount $|\Omega|$ of a given material $j$ is placed in order to resist to a given load $F$ ($F$ balanced, $|\text{supp}(F)| = 0$)
Shape optimization

Known: $Q$, $|\Omega|$, $j$, $F \in H^{-1}(Q)$, Unknown: the set (shape) $\Omega$

Goal:

$$\inf_{\Omega} \{ C(\Omega, j, F); \ |\Omega| \text{ fixed, } \Omega \subset Q \}$$
Shape optimization

Known: \( Q, |\Omega|, j, F \in H^{-1}(Q) \), Unknown: the set (shape) \( \Omega \)

Goal:

\[
\inf_{\Omega} \{ C(\Omega, j, F) ; \ |\Omega| \text{ fixed, } \Omega \subset Q \}
\]

-> no existence in general.
-> need of a relaxed formulation.

\[
C(\Omega, j, F) = \sup_u \{ < F, u > - \int j(e(u))1_\Omega(x) \, dx \}
\]

\[
C(\theta, j, F) = \sup_u \{ < F, u > - \int j(e(u))\theta(x) \, dx \}
\]

Find an optimal \( \theta \in L^\infty(Q, [0, 1]) \) instead of a characteristic function.
Shape optimization

-> new problem admits a solution $\bar{\theta}$ (fictitious material)
-> $0 < \bar{\theta} < 1$ needs an interpretation...
-> The infimum is not the same as in the original problem !!!

In the homogenized zones (where $0 < \theta < 1$ the resistance is weaker than $\theta j(e(u))$.
Should be replaced by $j^{hom}(\theta, e(u), structure)$ ...(Allaire)
The considered problem

The optimal shape in an asymptotically thin layer.

\[ \inf_{\Omega} \{ C(\Omega, j, F^\delta); \; \Omega \subset Q_\delta, \; |\Omega| = \tau \delta \} \]

No topological constraint (voids are allowed). It is different from optimizing plates thickness (Bonnetier, Conca)
The considered problem

The optimal shape in an asymptotically thin layer.

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First let us get rid of the volume constraint (using a Lagrange multiplier).

\[ \Phi^\delta(k) = \inf_{\Omega} \left\{ C(\Omega, j, F^\delta) + \frac{k}{\delta} |\Omega|; \; \Omega \subset Q_\delta \right\} \]

For a given \( k \geq 0 \) any solution of the new pb. is a solution of the original one. \(|\Omega|\) can be tuned by tuning \( k \). Both problems are equivalent.
Rescaling

-> Same scaling for $x, u, F^\delta$ as in plate theory $\rightarrow y, v, F$.

-> Same definition for $e^\delta(v)$.

-> Introduction of $\omega := \{(x_\alpha, \frac{1}{\delta}x_3); \; x \in \Omega\}$
Rescaling

-> Same scaling for $x, u, F^\delta$ as in plate theory -> $y, v, F$.

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\[
\Phi^\delta(k) = \inf_{\omega} \left\{ C^\delta(\omega) + \frac{k}{\delta} |\omega|; \ \Omega \subset Q_1 \right\}
\]

\[
C^\delta(\omega) := \sup_u \left\{ \langle F, u \rangle - \int_{Q_1} j(e^\delta(u)) 1_\omega(x) \, dx \right\}
\]
Rescaling

-> Same scaling for \( x, u, F^\delta \) as in plate theory \( \rightarrow y, v, F \).

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\Phi^\delta(k) = \inf_\omega \left\{ C^\delta(\omega) + \frac{k}{\delta}|\omega|; \quad \Omega \subset Q_1 \right\}
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\[
C^\delta(\omega) := \sup_u \left\{ <F, u> - \int_{Q_1} j(e^\delta(u)) 1_\omega(x) \, dx \right\}
\]

or in a dual way

\[
C^\delta(\omega) = \inf_\sigma \left\{ \int_{Q_1} j^*(\Pi^\delta(\sigma)); \quad \sigma = 0 \text{ on } Q_1 \setminus \omega, \quad \text{div}\sigma + F = 0 \right\}
\]

\[
\Pi^\delta(\sigma) := \begin{pmatrix}
\sigma_{\alpha,\beta} & \delta\sigma_{\alpha,3} \\
\delta\sigma_{\alpha,3} & \delta^2\sigma_{3,3}
\end{pmatrix}
\]
Remark

Indeed

\[ \int j^*(\Pi^\delta(\sigma)) + j(e^\delta(u)) \geq \int \Pi^\delta(\sigma) \cdot e^\delta(u) = \int \sigma \cdot e(u) = - \langle F, u \rangle \]
Description of the limit problem

Let \( \bar{j} \) be the energy in terms of displacement for plane stress.

\[
\bar{j}(A) := \inf_{\alpha} j\left( \begin{pmatrix} A & a_1 \\ a_1 & a_2 \\ a_1 & a_2 & a_3 \end{pmatrix} \right)
\]
Let $\overline{j}$ be the energy in term of displacement for plane stress.

$$\overline{j}(A) := \inf_a j \left( \begin{pmatrix} A & a_1 \\ a_1 & a_2 \end{pmatrix} \right)$$

The limit energy is described in term of the density $\theta$ of material in $Q_1$ (as in the fictitious material case).

$$C(\theta) = \sup_{u \in KL} \left\{ <F, u> - \int_{Q_1} \overline{j}(e_{\alpha,\beta}(u)) \theta(x) \, dx \right\}$$
Description of the limit problem

or in a dual way

\[ C(\theta) := \inf_{\sigma} \int_{Q_1} \theta^{-1} \times (\bar{f})^*(\sigma), \ div_\alpha[[\sigma]] + F_\alpha = 0, \ div_\alpha div_a[[y_3\sigma]] + F_3 = 0 \]

where \( \sigma \) is a \( 2 \times 2 \) matrix and \([[]]\) means the mean value on a section.

\[ [[\phi]](y_\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(y_\alpha, y_3) \, dy_3 \]
Description of the limit problem

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\[ C(\theta) := \inf_{\sigma} \left\{ \int_{Q_1} \theta^{-1} \times (\overline{f})^* (\sigma), \ \text{div}_\alpha [[\sigma]] + \overline{F}_\alpha = 0, \ \text{div}_\alpha \text{div}_a [[y_3 \sigma]] + \overline{F}_3 = 0 \right\} \]

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[[\phi]](y_\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(y_\alpha, y_3) \, dy_3
\]

Indeed

\[
\int \theta^{-1} \times (\overline{f})^* (\sigma) + (\overline{f})(e_{\alpha,\beta}(u) \theta - < F, u >
\]

\[
= \int \theta^{-1} \times ((\overline{f})^* (\sigma) + (\overline{f})(\theta e_{\alpha,\beta}(u))) - < F, u >
\]

\[
\geq \int \sigma \cdot e_{\alpha,\beta}(u) - < F, u > = \int -\text{div}_\alpha (\sigma) \cdot u - < F, u >
\]

which vanishes if \( \text{div}_\alpha (\sigma) + F \) is orthogonal to \( KL... \)
Description of the limit problem

The limit problem reads

\[ \Phi(k) = \inf_{\theta} \{ C'(\theta) + k \int_{Q_1} \theta \} \]

\[ \Phi(k) = \inf_{\theta} \sup_{u \in KL} \{ < F, u > - \int_{Q_1} \bar{j}(e(u))\theta + k \int_{Q_1} \theta \} \]
Description of the limit problem

The limit problem reads

$$\Phi(k) = \inf_\theta \{ C'(\theta) + k \int_{Q_1} \theta \}$$

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Theorem:

(i) the limit problem admits a solution.

(ii) $$\lim_{\delta \to 0} \Phi^\delta(k) = \Phi(k)$$

(iii) any minimizing sequence $$\omega^\delta$$ converges weakly* to a solution $$\overline{\theta}$$ of the limit pb.

(iv) the limit problem admits the following equivalent formulations.
Equivalent formulations

\[ \Phi(k) = \inf_{\theta} \sup_{u \in KL} \{ \langle F, u \rangle - \int_{Q_1} \bar{j}(e(u))\theta + k \int_{Q_1} \theta \} \]
Equivalent formulations

\[ \Phi(k) = \inf_{\theta} \sup_{u \in KL} \{ < F, u > - \int_{Q_1} \bar{j}(e(u))\theta + k \int_{Q_1} \theta \} \]

Inverting \( \inf \) and \( \sup \) and computing explicitly the infimum in \( \theta \) leads to the 3D formulation in terms of displacement only:

\[ \Phi(k) = \sup_{u \in KL} \{ < F, u > - \int_{Q_1} (\bar{j}(e(u)) - k)_+ \} \]
Equivalent formulations

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\[ \Phi(k) = \sup_{u \in KL} \{ < F, u > - \int_{Q_1} (\bar{j}(e(u)) - k)_+ \} \]

or 3D formulation in terms of stress only

\[ \Phi(k) = \inf_{\sigma} \{ \int_{Q_1} ((\bar{j} - k)_+)^*(\sigma); \text{div}_\alpha[[\sigma]] + \overline{F}_\alpha = 0, \text{div}_\alpha \text{div}_a[[x_3 \sigma]] + \overline{F}_3 = 0 \} \]
As $w$ is in $KL$, it can be explicit in terms of a 2D displacement $w$:

$$
\Phi(k) = \sup_w \{ <F, w> - \int_D W_k(e_{\alpha,\beta}(w_\alpha), \nabla^2_{\alpha,\beta}(w_3)) \}
$$

$$
W_k(A, B) := \int_{-\frac{1}{2}}^{\frac{1}{2}} (f(A - x_3 B) - k)_+
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Equivalent formulations

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\]

\[
W_k(A, B) := \int_{-\frac{1}{2}}^{\frac{1}{2}} (J(A - x_3 B) - k)_+
\]

The dual form of the last pb. involves 2D-generalized stresses.

\[
\Phi(k) = \inf_{\lambda, \eta} \int_D W^*_k(\lambda, \eta); \text{div}\lambda + F_\alpha = 0; \text{divdiv}\eta + F_3 = 0
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Equivalent formulations

As \( u \) is in \( KL \), it can be explicited in terms of a 2D displacement \( w \):

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\Phi(k) = \sup_{w} \{ <F, w> - \int_{D} W_{k}(e_{\alpha,\beta}(w_{\alpha}), \nabla^{2}_{\alpha,\beta}(w_{3})) \}
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W_{k}(A, B) := \int_{-\frac{1}{2}}^{\frac{1}{2}} (f(A - x_{3}B) - k)_{+}
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\]

(v) all these formulations have solutions \((\bar{u}, \bar{\sigma}, \bar{w}, \bar{\lambda}, \bar{\eta})\).

(vi) The optimal solutions correspond.

(vii) for any minimizing sequence, \( \Pi^{\delta}(\sigma^{\delta}(\omega^{\delta})) \) converges weakly* to \( \Pi^{0}(\bar{\sigma}) \).
Optimality conditions

\[ \bar{u} \in KL. \]
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- \( \text{div}_\alpha[[\overline{\sigma}]] + \overline{F}_\alpha = 0, \text{div}_\alpha\text{div}_a[[x_3\overline{\sigma}]] + \overline{F}_3 = 0 \)
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- $\bar{u} \in KL$.
- $\text{div}_\alpha[[\bar{\sigma}]] + \bar{F}_\alpha = 0$, $\text{div}_\alpha \text{div}_a[[x_3\bar{\sigma}]] + \bar{F}_3 = 0$
- $\bar{\sigma} = \bar{\theta} \frac{\partial j}{\partial e_{\alpha,\beta}}(e_{\alpha,\beta}(\bar{u}))$
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- $\overline{\sigma} = \overline{\theta} \frac{\partial j}{\partial e_{\alpha,\beta}}(e_{\alpha,\beta}(\overline{u}))$
- $\overline{\theta} \times (\overline{j}(e_{\alpha,\beta}(\overline{u}) - k) = (\overline{j}(e_{\alpha,\beta}(\overline{u}) - k)_+$
Optimality conditions

- \( \bar{u} \in KL \).
- \( \text{div}_\alpha[[\bar{\sigma}]] + \bar{F}_\alpha = 0, \text{div}_\alpha \text{div}_a[[x_3\bar{\sigma}]] + \bar{F}_3 = 0 \)
- \( \bar{\sigma} = \bar{\theta} \frac{\partial \tilde{j}}{\partial e_{\alpha,\beta}}(e_{\alpha,\beta}(\bar{u})) \)
- \( \bar{\theta} \times (\tilde{j}(e_{\alpha,\beta}(\bar{u}) - k) = (\tilde{j}(e_{\alpha,\beta}(\bar{u}) - k) + \)

But \( e_{\alpha,\beta}(\bar{u}) = e_{\alpha,\beta}(\bar{w}) + \frac{\partial^2 w_3}{\partial x_\alpha \partial x_\beta} \times x_3 \) is linear

![Graphs showing optimality conditions and linear behavior](image-url)
Optimal structures

Assume that $\frac{\partial^2 w_3}{\partial x_\alpha \partial x_\beta} \neq 0$ (generic case).

Then, at point $y_\alpha$, $\theta(y_\alpha, y_3)$ is the characteristic function of the complementary of an interval.

The optimal structure is a real shape made of two plates with varying thickness along the top and bottom of the design domain.
From the mechanical point of view:
- concentrating the material on the top and the bottom is usual (T-shape or I-shape for beam, sandwishes-shape for plates)
- but using two disconnected plates seems idiot.
Optimal structures

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Optimal structures

The point is that a negligible amount of material is needed for connecting the two plates (it disappears at the limit).

- The same is true for the boundary layers needed for taking into account the applied forces.

Note that the relative thickness of the two layers depends on $e_{\alpha,\beta}(\bar{w})$. 
Optimal structures

The thickness depends on $x_\alpha$ through $\frac{\partial^2 w_3}{\partial x_{\alpha} \partial x_{\beta}}$ which varies from place to place.

Example of the “bridge”:

Note that the dark region is not given by the limit problem. It’s a “decoration”. The fact that it is a higher order problem likely explains why there are so many different shapes used for this intermediate part.
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