3D-2D Analysis for the optimal elastic compliance problem

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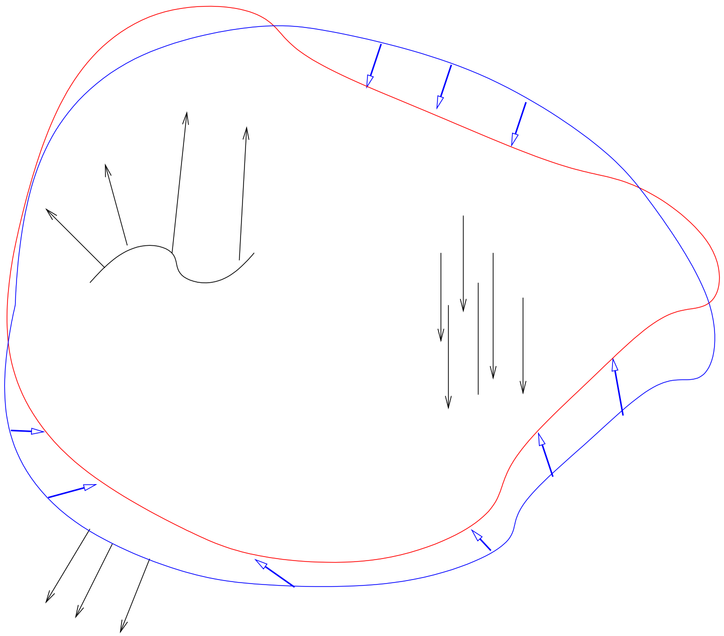
The framework of linear elasticity



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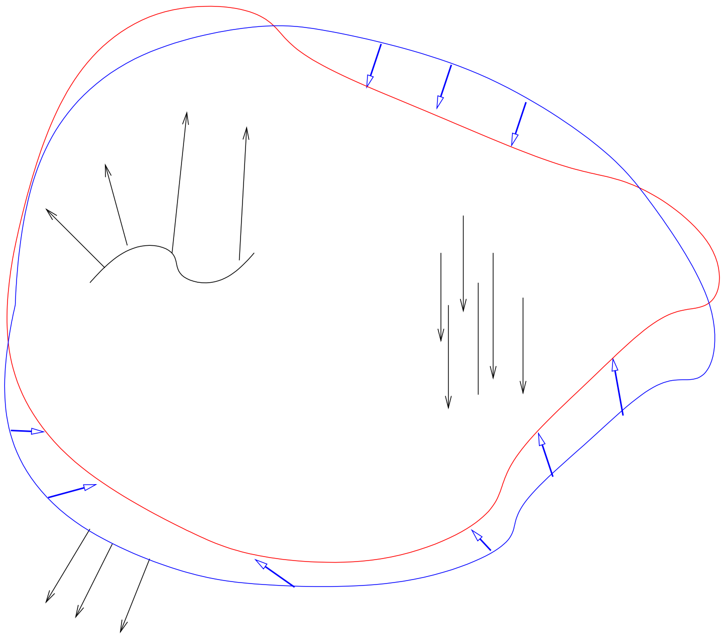

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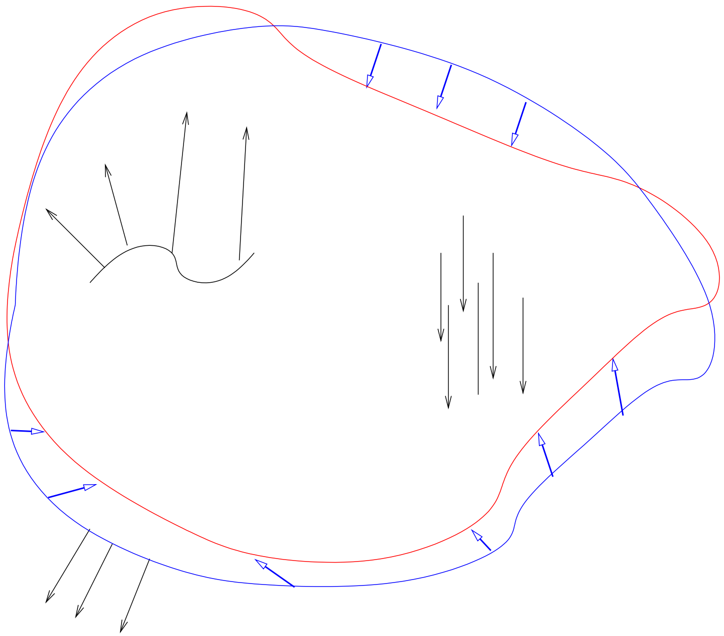

- A displacement field u on Ω
- The (linearized) strain tensor field $e(u) = (\nabla u + \nabla u^T)/2$
- A quadratic coercive energy density $j(e(u))$
  (e.g. $j(e(u)) = \lambda(tr(e(u))^2 + \mu/2\ \|e(u)\|^2$)

The framework of linear elasticity

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$$-C = \inf_u \{ \int_\Omega j(e(u)) - < F, u > \}$$

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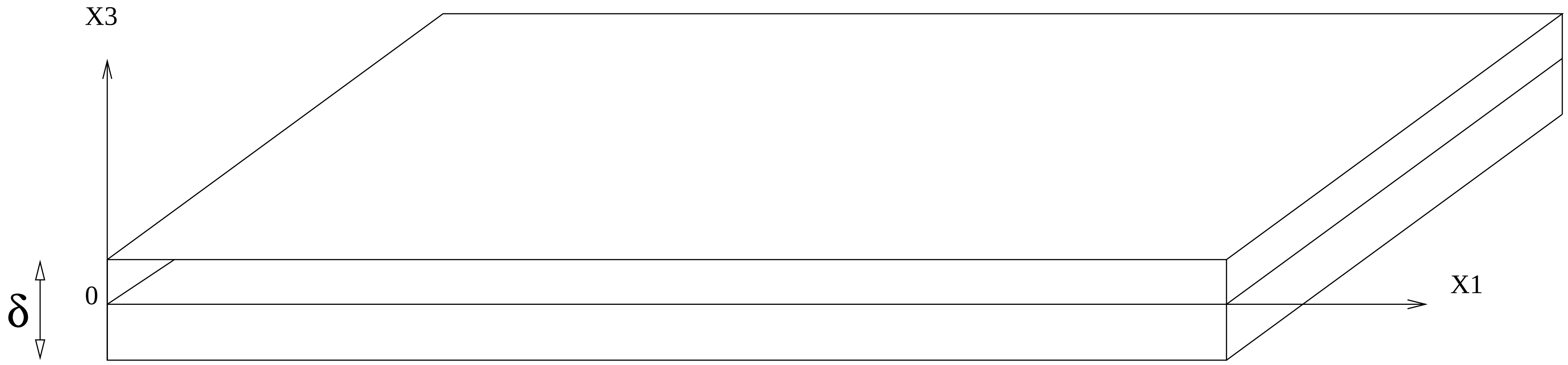
Dual formulation

$$C = \inf_{\Sigma} \{ \int_{\Omega} j^*(\Sigma);\ div(\Sigma) + F = 0 \}$$

$$j^*(A) := \sup_{B} \{ A \cdot B - j(B) \}$$

3D-2D asymptotic analysis



$$Q_\delta = \mathcal{D} \times [-\frac{\delta}{2}, \frac{\delta}{2}], \qquad \delta \to 0$$

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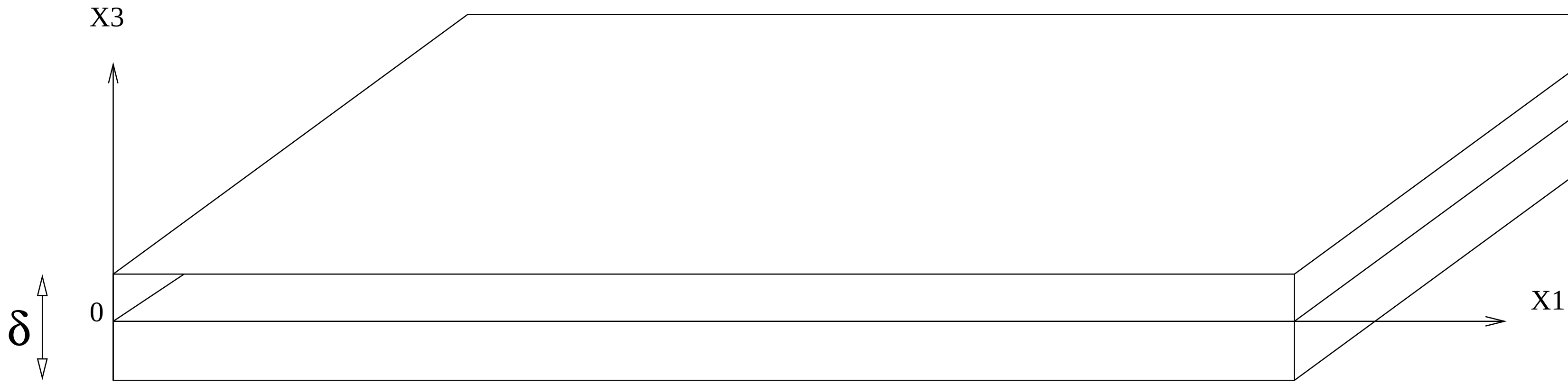

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Question :

$$\lim_{\delta \to 0} \inf_u \{ \int_{Q_\delta} j(e(u)) - < F^\delta, u > \}?$$

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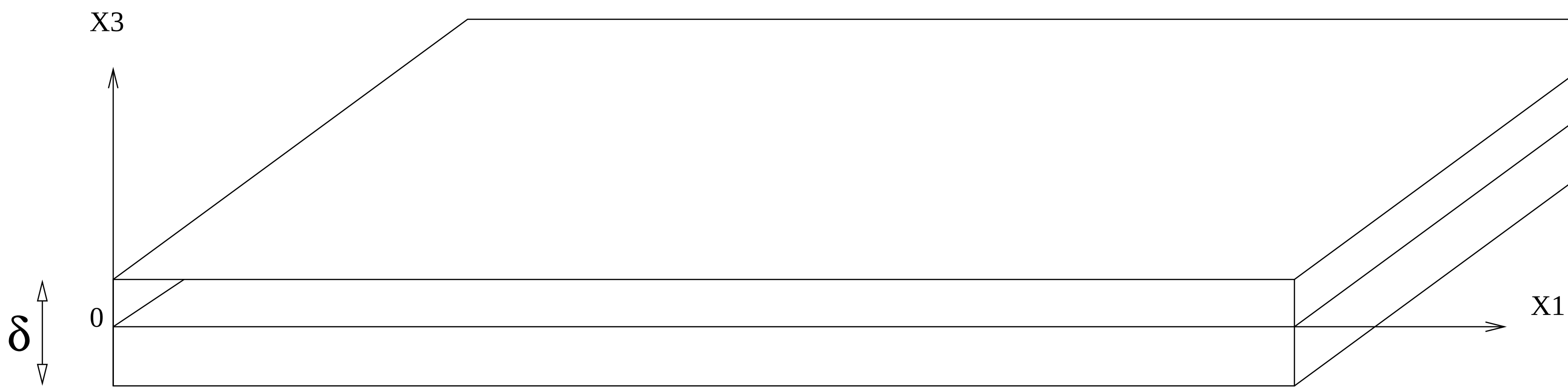

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Rescaling :

$$y_\alpha = x_\alpha \ (\alpha = 1,\ 2),\ y_3 = \frac{x_3}{\delta}$$

3D-2D asymptotic analysis

$$u_\alpha(x_\alpha, x_3) = v_\alpha(x_a, \frac{x_3}{\delta}),\ u_3(x_\alpha, x_3) = \frac{1}{\delta} v_3(x_a, \frac{x_3}{\delta}).$$

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$$e(u)(x_\alpha, x_3) = e^\delta(v) := \begin{pmatrix} e_{\alpha,\beta}(v) & \frac{1}{\delta} e_{\alpha,3}(v) \\ \frac{1}{\delta} e_{\alpha,3}(v) & \frac{1}{\delta^2} e_{3,3}(v) \end{pmatrix}$$

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$$F_\alpha^\delta(x_\alpha, x_3) = \frac{1}{\sqrt{\delta}} F_\alpha(x_a, \frac{x_3}{\delta}), F_3^\delta(x_\alpha, x_3) = \sqrt{\delta} F_3(x_a, \frac{x_3}{\delta}),$$

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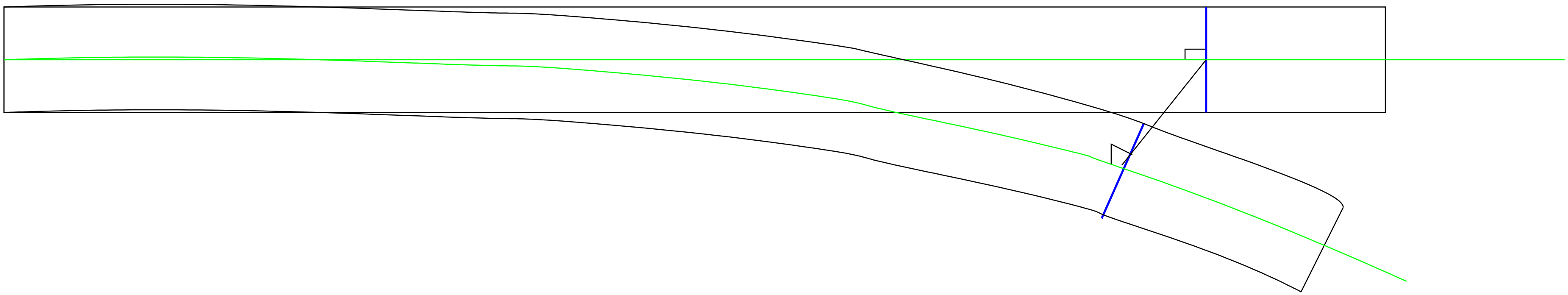
Rescaled problem :

$$\inf_v \{ \int_{Q_1} j(e^\delta(v)) - < F, v > \}$$

3D-2D asymptotic analysis

Definition : $KL := \{v \in H^1(Q_1)/e_{\alpha,3}(v) = 0,\ e_{3,3}(v) = 0\}$



Characterization : v is in KL if and only if there exists w such that $w_\alpha \in H^1(\mathcal{D})$, $w_3 \in H^2(\mathcal{D})$ and

$$v_3(y_\alpha, y_3) = w_3(y_\alpha),\ v_\alpha(y_\alpha, y_3) = w_\alpha(y_\alpha) - \frac{\partial w_3}{\partial y_\alpha} y_3$$

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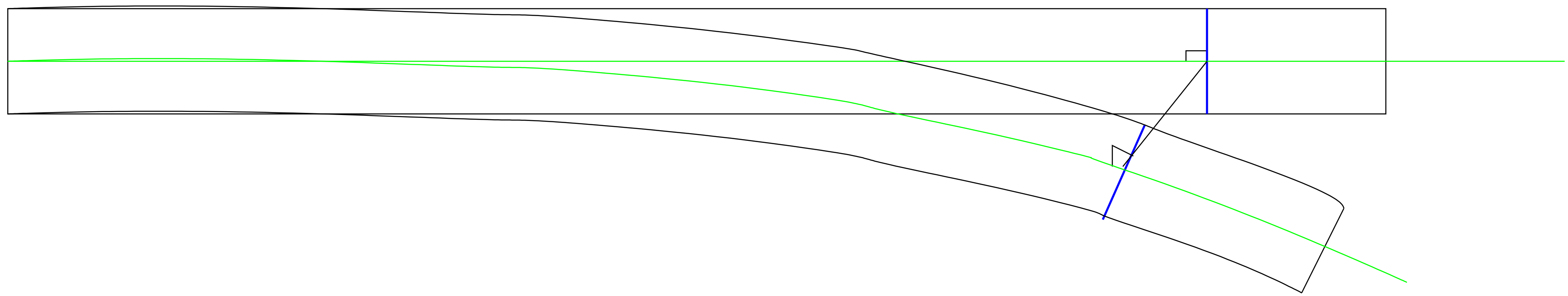

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- replace v by a KL-displacement in the infimum pb.
- explicit the energy in term of w

3D-2D asymptotic analysis

- gives the 2D plate problem :

$$\inf_{w}\{\int_{\mathcal{D}} \mathcal{W}(e_{\alpha,\beta}(w_\alpha), \nabla^2 w_3) - \overline{F}, w >\}$$

where

$$\overline{F}_\alpha(y_\alpha) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F_\alpha(y_\alpha, y_3)\, dy_3$$

$$\overline{F}_3(y_\alpha) := \int_{-\frac{1}{2}}^{\frac{1}{2}} F_3 + y_3 \frac{\partial F_1}{\partial y_1} + x_3 \frac{\partial F_2}{\partial y_2}\, dy_3$$

and $\mathcal{W}$ is explicit in terms of j.

3D-2D asymptotic analysis

-> many extensions (non isotropic, non homogeneous, varying thickness, non linear cases)

-> but no simple extension with high contrast composites (or with voids)

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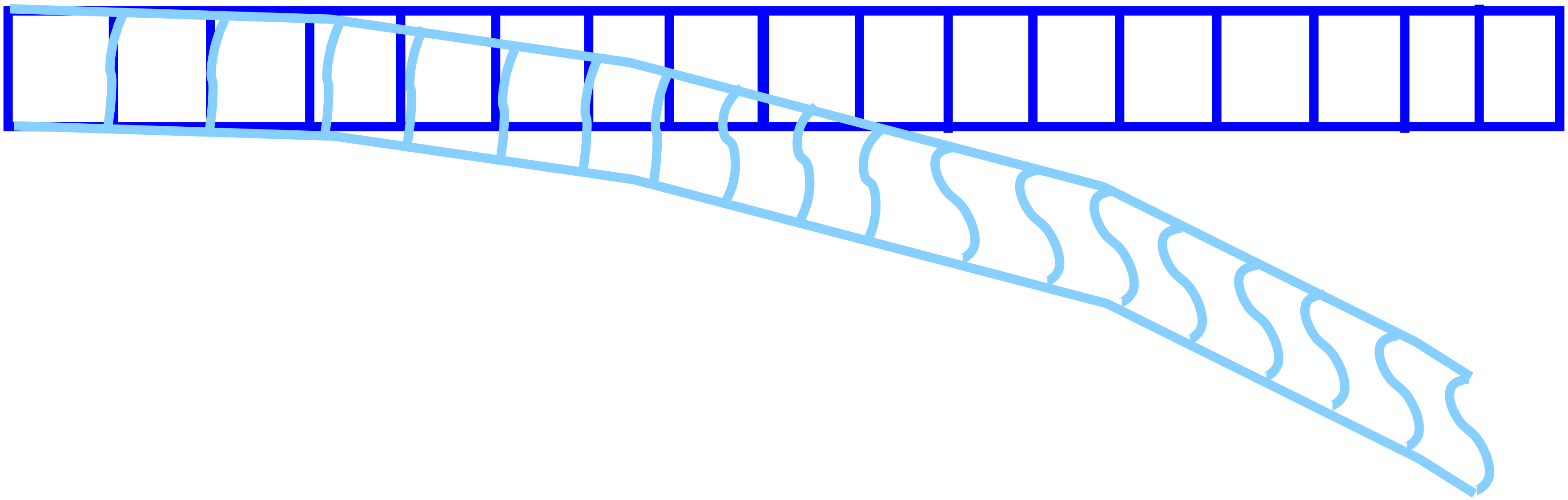
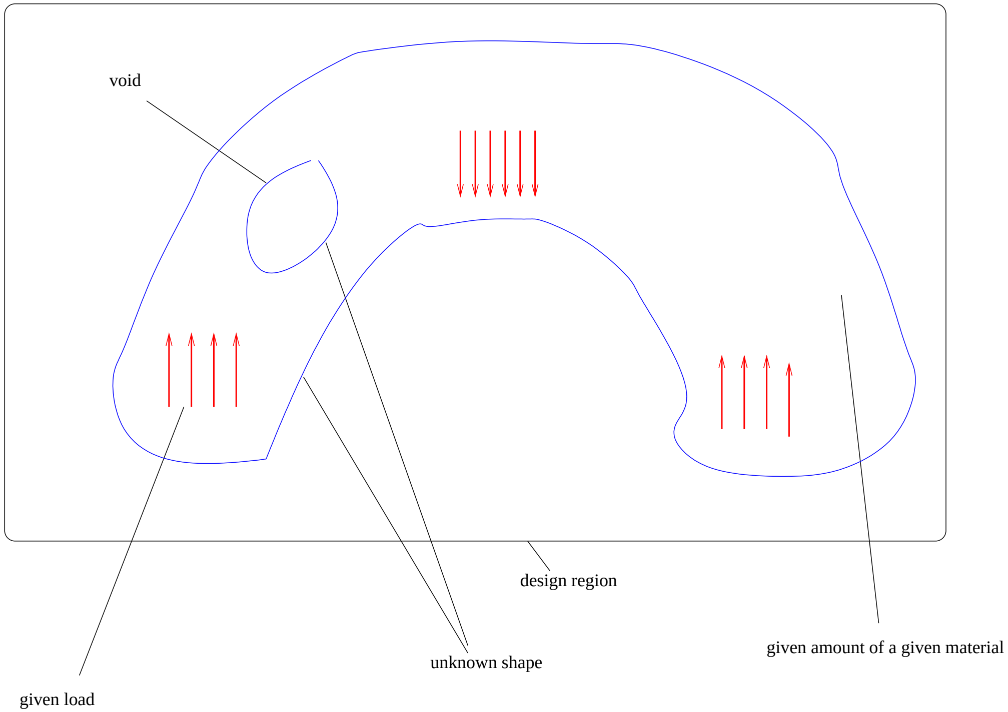


Shape optimization

In a given design region Q, a given amount $|\Omega|$ of a given material j is placed in order to resist to a given load F (F balanced, $|supp(F)| = 0$)



Shape optimization

Known : Q, $|\Omega|$, j, $F \in H^{-1}(Q)$, Unknown : the set (shape) Ω

Goal :

$$\inf_{\Omega}\{C(\Omega, j, F);\ |\Omega| \text{ fixed},\ \Omega \subset Q\}$$

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-> no existence in general.
-> need of a relaxed formulation.

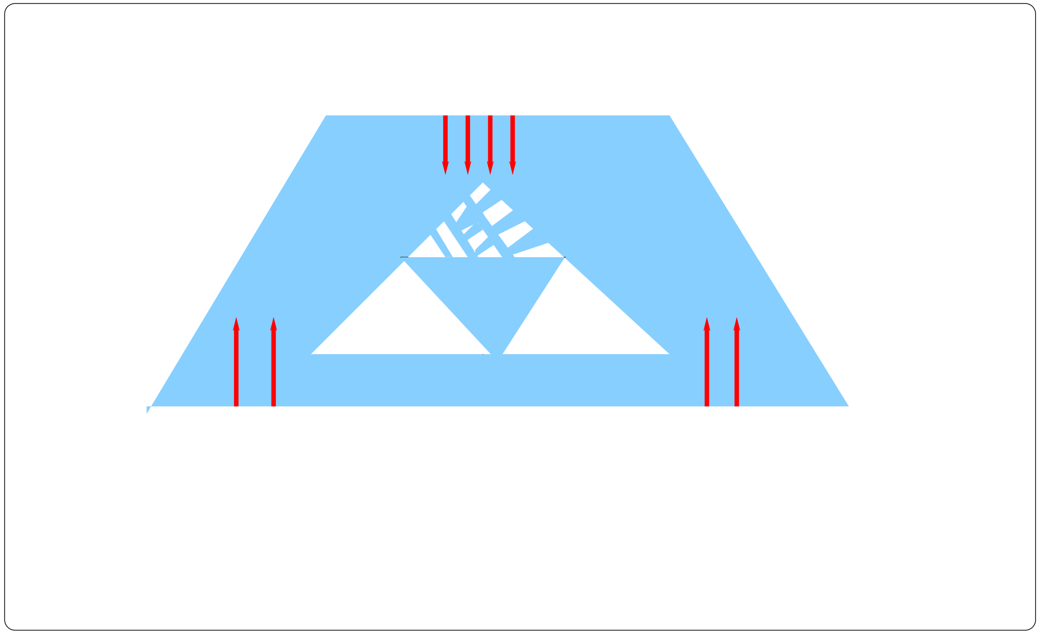
$$C(\Omega, j, F) = \sup_u \{< F, u > - \int j(e(u)) 1_{\Omega}(x)\, dx\}$$

$$C(\theta, j, F) = \sup_u \{< F, u > - \int j(e(u)) \theta(x)\, dx\}$$

Find an optimal $\theta \in L^{\infty}(Q, [0, 1])$ instead of a characteristic function.

Shape optimization

-> new problem admits a solution $\overline{\theta}$ (fictituous material)
-> $0 < \overline{\theta} < 1$ needs an interpretation...
-> The infimum is not the same as in the original problem !!!



In the homogenized zones (where $0 < \theta < 1$ the resistance is weaker than $\theta j(e(u))$.
Should be replaced by $j^{hom}(\theta, e(u), structure)$ ...(Allaire)

The considered problem

The optimal shape in an asymptotically thin layer.

$$\inf_{\Omega}\{C(\Omega, j, F^{\delta});\ \Omega \subset Q_{\delta},\ |\Omega| = \tau\delta\}$$

No topological constraint (voids are allowed). It is different from optimizing plates thickness (Bonnetier, Conca)

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First let us get rid of the volume constraint (using a Lagrange multiplier).

$$\Phi^{\delta}(k) = \inf_{\Omega}\{C(\Omega, j, F^{\delta}) + \frac{k}{\delta}|\Omega|;\ \Omega \subset Q_{\delta}\}$$

For a given $k \geq 0$ any solution of the new pb. is a solution of the original one. $|\Omega|$ can be tuned by tuning k. Both problems are equivalent.

Rescaling

-> Same scaling for x, u, F^δ as in plate theory -> y, v, F.

-> Same definition for $e^\delta(v)$.

-> Introduction of $\omega := \{(x_\alpha, \frac{1}{\delta}x_3);\ x \in \Omega\}$

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$$\Phi^\delta(k) = \inf_\omega \{C^\delta(\omega) + \frac{k}{\delta}|\omega|;\ \Omega \subset Q_1\}$$

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$$C^\delta(\omega) := \sup_u\{< F, u > - \int_{Q_1} j(e^\delta(u))\, 1_\omega(x)\, dx$$

or in a dual way

$$C^\delta(\omega) = \inf_\sigma\{\int_{Q_1} j^*(\Pi^\delta(\sigma));\ \sigma = 0 \text{ on } Q_1 \setminus \omega,\ div\sigma + F = 0\}$$

$$\Pi^\delta(\sigma) := \begin{pmatrix} \sigma_{\alpha,\beta} & \delta\sigma_{\alpha,3} \\ \delta\sigma_{\alpha,3} & \delta^2\sigma_{3,3} \end{pmatrix}$$

Remark

Indeed

$$\int j^*(\Pi^\delta(\sigma)) + j(e^\delta(u)) \geq \int \Pi^\delta(\sigma) \cdot e^\delta(u) = \int \sigma \cdot e(u) = - < F, u >$$

Description of the limit problem

Let $\overline{j}$ be the energy in term of displacement for plane stress.

$$\overline{j}(A) := \inf_a j\left(\begin{pmatrix} & A & & a_1 \\ & & & a_2 \\ a_1 & & a_2 & a_3 \end{pmatrix}\right)$$

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The limit energy is described in term of the density θ of material in Q_1 (as in the fictituous material case).

$$C(\theta) = \sup_{u \in KL} \{< F, u > - \int_{Q_1} \bar{j}(e_{\alpha,\beta}(u))\, \theta(x)\, dx\}$$

Description of the limit problem

or in a dual way

$$C(\theta) := \inf_{\sigma}\{\int_{Q_1} \theta^{-1} \times (\bar{j})^*(\sigma),\ div_\alpha[[\sigma]] + \overline{F}_\alpha = 0,\ div_\alpha div_a[[y_3\sigma]] + \overline{F}_3 = 0\}$$

where σ is a 2×2 matrix and $[[.]]$ means the mean value on a section.

$$[[\phi]](y_\alpha) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(y_\alpha, y_3)\, dy_3$$

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Indeed

$$\int \theta^{-1} \times (\bar{j})^*(\sigma) + (\bar{j})(e_{\alpha,\beta}(u)\theta - < F, u >$$

$$= \int \theta^{-1} \times \left((\bar{j})^*(\sigma) + (\bar{j})(\theta\, e_{\alpha,\beta}(u)\right) - < F, u >$$

$$\geq \int \sigma \cdot e_{\alpha,\beta}(u) - < F, u >= \int -div_\alpha(\sigma) \cdot u - < F, u >$$

which vanishes if $div_\alpha(\sigma) + F$ is orthogonal to KL...

Description of the limit problem

The limit problem reads

$$\Phi(k) = \inf_{\theta} \{ C(\theta) + k \int_{Q_1} \theta \}$$

$$\Phi(k) = \inf_{\theta} \sup_{u \in KL} \{ < F, u > - \int_{Q_1} \overline{j}(e(u))\theta + k \int_{Q_1} \theta \}$$

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Theorem :

(i) the limit problem admits a solution.

(ii) $lim_{\delta\to 0}\Phi^{\delta}(k) = \Phi(k)$

(iii) any minimizing sequence ω^{δ} *converges weakly* to a solution* $\overline{\theta}$ *of the limit pb.*

(iv) the limit problem admits the following equivalent formulations.

Equivalent formulations

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Inverting $\inf$ *and* $\sup$ *and computing explicitely the infimum in* θ *leads to the 3D formulation in terms of displacement only :*

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or 3D formulation in terms of stress only

$$\Phi(k) = \inf_{\sigma} \{\int_{Q_1} ((\bar{j} - k)_+)^*(\sigma);\ div_\alpha[[\sigma]] + \overline{F}_\alpha = 0,\ div_\alpha div_a[[x_3\sigma]] + \overline{F}_3 = 0\}$$

Equivalent formulations

As u *is in* KL*, it can be explicited in terms of a 2D displacement* w *:*

$$\Phi(k) = \sup_{w}\{< \overline{F}, w > - \int_{\mathcal{D}} W_k(e_{\alpha,\beta}(w_\alpha), \nabla^2_{\alpha,\beta}(w_3))$$

$$W_k(A, B) := \int_{-\frac{1}{2}}^{\frac{1}{2}} (\bar{j}(A - x_3 B) - k)_+$$

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The dual form of the last pb. involves 2D-generalized stresses.

$$\Phi(k) = \inf_{\lambda,\eta}\{\int_{\mathcal{D}} W_k^*(\lambda, \eta); div\lambda + \overline{F}_\alpha = 0;\ divdiv\eta + \overline{F}_3 = 0\}$$

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(v) all these formulations have solutions $(\overline{u}, \overline{\sigma}, \overline{w}, \overline{\lambda}, \overline{\eta})$.
(vi) The optimal solutions correspond.
(vii) for any minimizing sequence, $\Pi^\delta(\sigma^\delta(\omega^\delta))$ *converges weakly* to* $\Pi^0(\overline{\sigma})$.

Optimality conditions

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- $\overline{\sigma} = \overline{\theta}\frac{\partial \overline{j}}{\partial e_{\alpha,\beta}}(e_{\alpha,\beta}(\overline{u}))$

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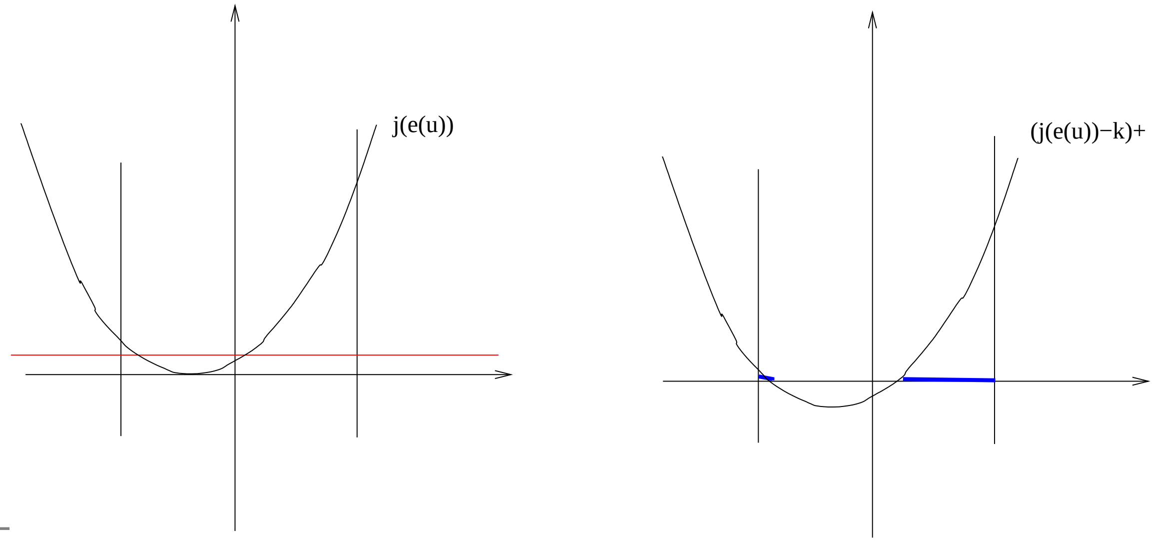
- $\overline{\theta} \times (\overline{j}(e_{\alpha,\beta}(\overline{u}) - k) = (\overline{j}(e_{\alpha,\beta}(\overline{u}) - k)_+$

But $e_{\alpha,\beta}(\overline{u}) = e_{\alpha,\beta}(\overline{w}) + \frac{\partial^2 \overline{w}_3}{\partial x_\alpha \partial x_\beta} \times x_3$ is linear

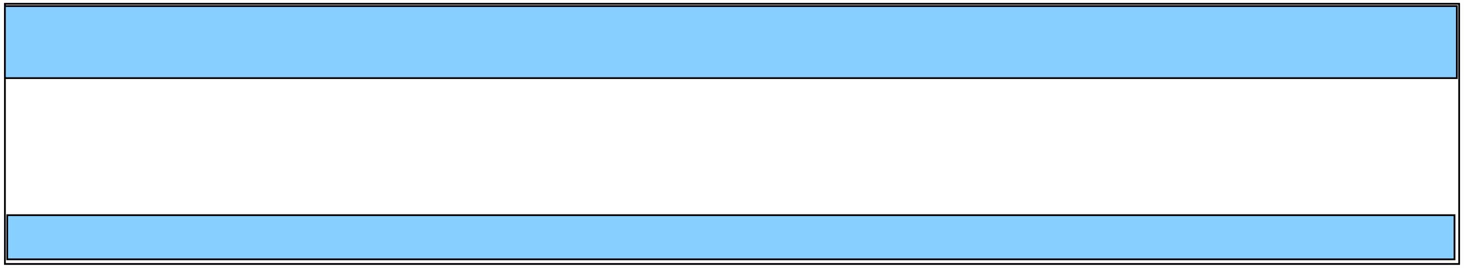


Optimal structures

Assume that $\frac{\partial^2 \overline{w}_3}{\partial x_\alpha \partial x_\beta} \neq 0$ (generic case).

Then, at point y_α, $\theta(y_\alpha, y_3)$ is the characteristic function of the complementary of an interval..

The optimal structure is a real shape made of two plates with varying thickness along the top and bottom of the design domain.



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- concentrating the material on the top and the bottom is usual (T-shape or I-shape for beam, sandwishes-shape for plates)

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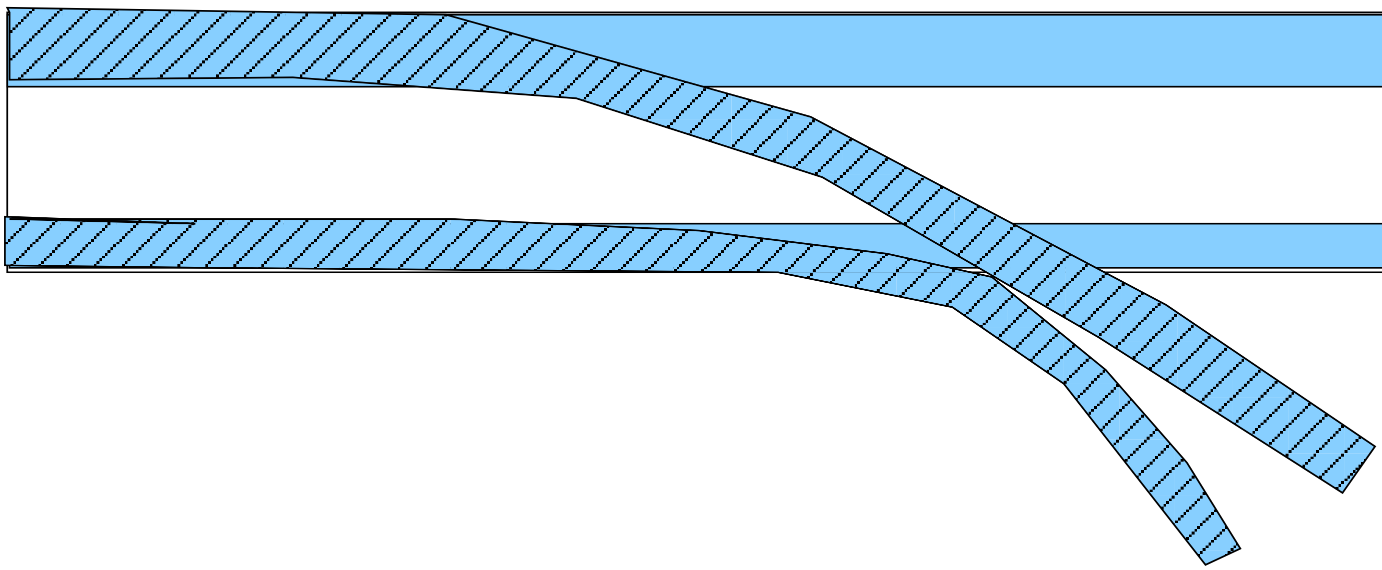
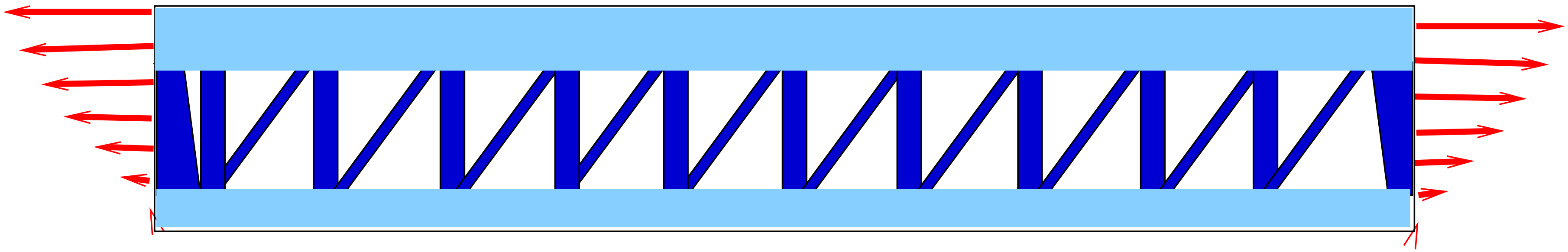


Optimal structures

The point is that a negligible amount of material is needed for connecting the two plates (it disappears at the limit).

- The same is true for the boundary layers needed for taking into account the applied forces.

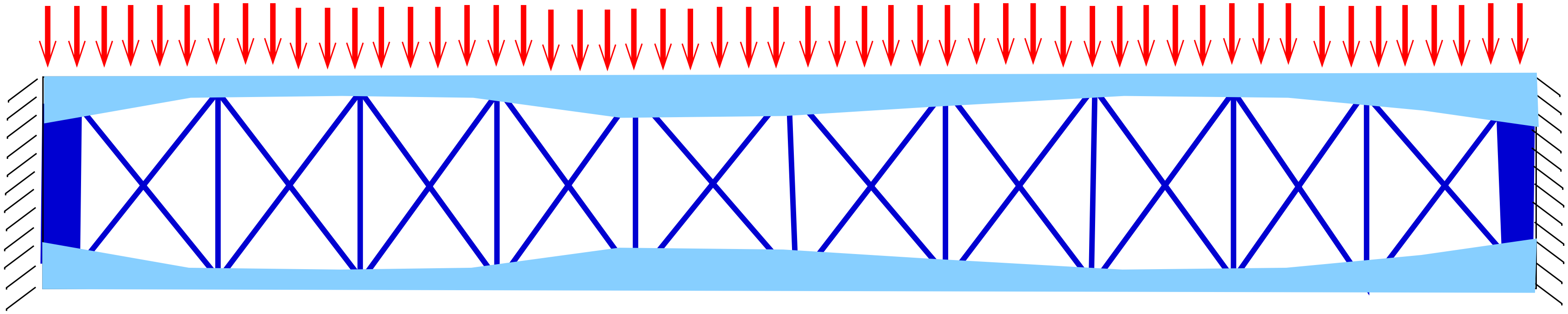


Note that the relative thickness of the two layers depends on $e_{\alpha,\beta}(\overline{w})$.

Optimal structures

The thickness depends on x_α through $\frac{\partial^2 \overline{w}_3}{\partial x_\alpha \partial x_\beta}$ which varies from place to place.

Example of the “bridge” :



Note that the dark region is not given by the limit problem. It’s a “decoration”.
The fact that it is a higher order problem likely explains why there are so many different shapes used for this intermediate part.

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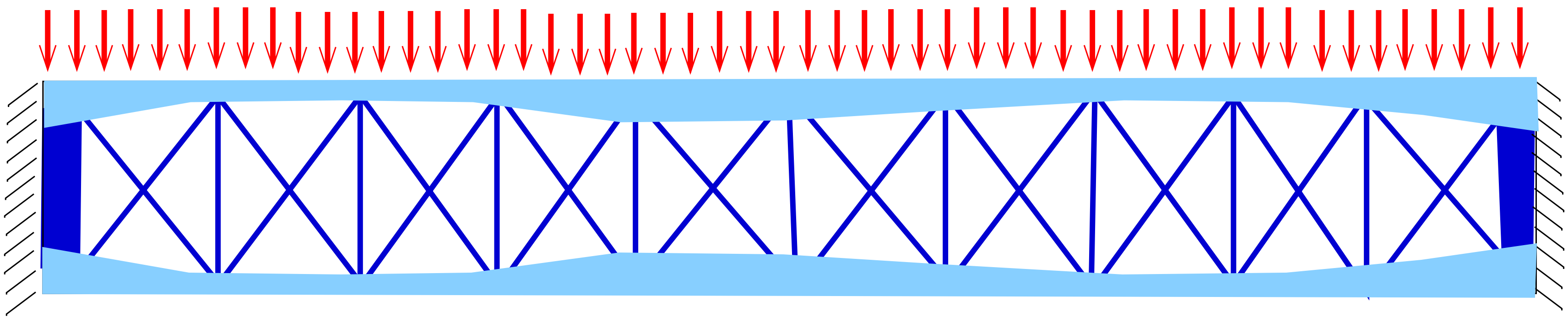

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