Evolution equations and Tikhonov regularization

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Outline

1. Generalized gradient method and Tikhonov regularization
2. Coupling evolution and regularization in convex minimization: strong and weak convergence
3. The case of differential inclusions governed by maximal monotone operators

Generalized gradient method

Let f be a proper lower-semicontinuous convex function on a Hilbert space H and let

$$\mathcal{S} = \operatorname{Argmin} f.$$

Functions u satisfying

$$-\dot{u}(t) \in \partial f(u(t))$$

are minimizing. Moreover, if $\mathcal{S} \neq \emptyset$ they converge weakly as $t \to \infty$ to some $x_\infty \in \mathcal{S}$.

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Tikhonov regularization

For $\varepsilon > 0$ consider the strongly convex function

$$f_\varepsilon(x) = f(x) + \frac{\varepsilon}{2}\|x\|^2.$$

The solutions of

$$-\dot{u}(t) \in \partial f_\varepsilon(u(t))$$

converge strongly as $t \to \infty$ to x_ε, the unique minimizer of f_ε.
If $\mathcal{S} \neq \emptyset$ then x_ε converges strongly to the least-norm element x^* of $\mathcal{S}$.

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Coupling Tikhonov regularization and the gradient method

Coupling

Let ε be a positive function on $[0, \infty)$ such that $\lim_{t \to \infty} \varepsilon(t) \to 0$ and let $u : [0, \infty) \to \mathcal{H}$ satisfy

$$-\dot{u}(t) \in \partial f_{\varepsilon(t)}(u(t)) = \partial f(u(t)) + \varepsilon(t)u(t).$$

Theorem (Cominetti, P. & Sorin)

(i) *If* $\int_0^\infty \varepsilon(t)\, dt = \infty$ *then* $u(t) \to x^*$.

(ii) *If* $\int_0^\infty \varepsilon(t)\, dt < \infty$ *then* $u(t) \rightharpoonup x_\infty$ *for some* $x_\infty \in S$.

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Proof

(i) Let $\theta(t) = \frac{1}{2}|u(t) - x^*|^2$. One easily proves

$$\dot{\theta}(t) + \varepsilon(t)\theta(t) \leq \tfrac{1}{2}\varepsilon(t)\left[|x^*|^2 - |x_{\varepsilon(t)}|^2\right].$$

A Gronwall-like inequality then gives

$$0 \leq \limsup_{t\to\infty} \theta(t) \leq \limsup_{t\to\infty} \frac{1}{2}\left[|x^*|^2 - |x_{\varepsilon(t)}|^2\right] = 0.$$

(ii) Will be done later. ∎

Previous attempts

Under additional assumptions on ε and x_ε:

(i)

- Attouch-Cominetti 1996.
- Baillon-Cominetti 2001.
- Cabot 2004.
- Reich 1976.

(ii)

- Cominetti-Alemany 1999.
- Cabot 2004.
- Furuya-Miyashiba-Kenmochi 1986.
- Alvarez-P. 2007

Tikhonov regularization and differential inclusions governed by monotone operators

Monotone operators

A (possibly multi-valued) map $A : H \to 2^H$ is monotone if

$$\langle\, x^* - y^*, x - y \,\rangle \geq 0 \quad \text{for all} \quad y \in Ax\,, y^* \in Ax^*$$

and maximal if its graph is not properly contained in the graph of any other monotone operator.

Examples

Examples:

- $A = \partial f$; with f closed, proper and convex.
- $A = I - F$; with F nonexpansive.

The solution set

$$S = A^{-1}0$$

coincides with

$$\text{the minimizers of } f \quad \text{if} \quad A = \partial f$$

$$\text{the fixed points of } F \quad \text{if} \quad A = I - F$$

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Asymptotic behavior of $-\dot{u} \in Au$

In general, $\mathcal{S} \neq \emptyset$ does not imply that functions satisfying $-\dot{u}(t) \in Au(t)$ converge (although it does imply they converge *in average*).

Counterexample

For
$$A(x, y) = (-y, x)$$
one gets
$$u(t) = r_0 \left(\cos(2t_0 - t), \sin(2t_0 - t)\right),$$
which does not converge unless $r_0 = 0$.

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Tikhonov regularization

In what follows we assume ε is a positive function on $[0, \infty)$ such that $\lim_{t \to \infty} \varepsilon(t) \to 0$ and study the behavior as $t \to \infty$ of functions $u : [0, \infty) \to \mathcal{H}$ satisfying

$$-\dot{u}(t) \in Au(t) + \varepsilon(t)u(t)$$

according to whether the function ε is in L^1 or not.

$\varepsilon \in L^1$

Theorem

A perturbation $\varepsilon \in L^1(0, \infty; \mathbb{R})$ *makes no difference in the asymptotic behavior of the system.*

There is no loss – and no gain! – in applying the Tikhonov regularization in this case.

Proof I

Let $A(\cdot)$ be a family of maximal monotone operators and let $\varepsilon \in L^1(0, \infty; \mathbb{R})$.

Lemma (Alvarez & P.)

If every function $u : [0, \infty) \to \mathcal{H}$ satisfying

$$-\dot{u}(t) \in A(t)u(t)$$

converges strongly (weakly) as $t \to \infty$, so does every function $v : [0, \infty) \to \mathcal{H}$ satisfying

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Proof II

Let $U(t,s)x = u(t)$, where

$$\begin{cases} -\dot{u}(t) & \in & Au(t) \\ u(s) & = & x \end{cases}$$

and let y satisfy

$$-y(t) \in Ay(t) + \varepsilon(t)y(t).$$

First on proves

$$\lim_{t\to\infty}\left[\sup_{h\geq 0}\|y(t+h) - U(t+h,t)y(t)\|\right] = 0.$$

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Proof III

Next,

$$\lim_{t\to\infty}\left[\tau - \lim_{h\to\infty} U(t+h,t)y(t)\right] = \zeta.$$

Finally one writes

$$y(t+h) - \zeta = [y(t+h) - U(t+h,t)y(t)] + [U(t+h,t)y(t) - \zeta]$$

and concludes that

$$\tau - \lim_{t\to\infty} y(t) = \zeta. \qquad \blacksquare$$

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$\varepsilon \notin L^1$ and bounded total variation

Assume $\mathcal{S} \neq \emptyset$ and $\varepsilon \notin L^1$.

Theorem (Cominetti, P. & Sorin)

If $\int_0^\infty |\dot{\varepsilon}(t)|\, dt < \infty$ then any function $u : [0, \infty) \to \mathcal{H}$ satisfying

$$-\dot{u}(t) \in Au(t) + \varepsilon(t)u(t)$$

converges strongly as $t \to \infty$ to the least-norm element of $\mathcal{S}$.

Idea of the proof

One first proves that

$$\lim_{t\to\infty} \dot{u}(t) = 0.$$[1]

Next we verify that, as a consequence,

$$\text{all weak cluster points of } u(t) \text{ for } t \to \infty \text{ belong to } S$$

Finally, the latter implies

$$u(t) \to x^* \text{ strongly.}$$ ∎

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The case of infinite total variation

If

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the solutions of

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The operator A

Let A be the $\frac{\pi}{2}$-counterclockwise rotation around $p = (1, 1)$; *i.e.*

$$A(x, y) = (2 - y, x).$$

We consider the system

$$-\dot{u}(t) = Au(t) + \varepsilon(t)u(t).$$

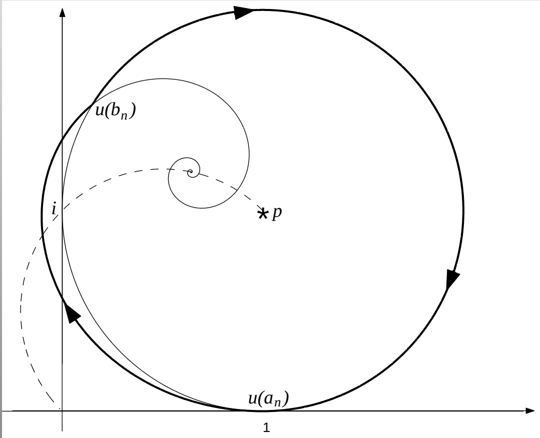
The parameter function ε

Let ε_n be a sequence of positive real numbers with $\varepsilon_n \to 0$ and $\sum \varepsilon_n = \infty$. Take $a_0 = 0$ and let $b_n = a_n + \tau_n$, $a_{n+1} = b_n + \sigma_n$ with $\tau_n > 0, \sigma_n > 0$ to be fixed later on, and consider the step function

$$\varepsilon(t) = \begin{cases} \varepsilon_n & \text{if} \quad a_n \leq t < b_n \\ 0 & \text{if} \quad b_n \leq t < a_{n+1}. \end{cases}$$

Clearly $\varepsilon(t) \to 0$ and we get $\int_0^\infty \varepsilon(t)dt = \infty$ provided τ_n is bounded away from zero.

Global behavior



Regularity

The lack of continuity of the function ε is not the problem, nor is it the fact that ε vanishes in some intervals.

In fact, one can find a strictly positive, infinitely differentiable function η such that $\eta \notin L^1$ while $\varepsilon - \eta \in L^1$.

The method also yields nonconvergent trajectories with this new parameter function.

Thank you

THANK YOU !