Kurdyka-Łojasiewicz inequality and subgradient trajectories : the convex case

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Łojasiewicz inequality

Łojasiewicz inequality [Łojasiewicz 1963]
$f:\mathbb{R}^N \to \mathbb{R}$ is **analytic**.
Let a be a critical point of f.
Then there exists a neighborhood U of a, $C>0$ and $\theta \in (0,1)$ such that

$$||\nabla f(x)|| \geq |f(x)-f(a)|^{\theta} \quad \text{for all } x \in U.$$

➪ Finite length of the gradient trajectories, Every critical point is limit of a gradient trajectory, etc.

Basic assumptions

To simplify :

H is a real Hilbert space
$f : H \to [0, +\infty)$ is smooth ($f \geq 0$)

For all $r > 0$, $C_r := \{f \leq r\}$

(A0) (0 is a critical point and a global minimum)
$0 \in C_0$

(A1) (0 is an isolated critical value)
There exits $r_0 > 0$ such that :
$x \in C_{r_0}$ and $f(x) > 0 \;\Rightarrow\; \nabla f(x) \neq 0$

(A2) (Sublevel compactness)
There exits $r_0 >$ such that : $C_{r_0} = \{f \leq r_0\}$ is compact.

Generalization : KŁ-inequality

We say that f satisfies Kurdyka-Łojasiewicz inequality [Kurdyka 1998] if :

There exists $\varphi \in KL(0, r_0)$ such that :

$$||\nabla(\varphi \circ f)(x)|| \geq 1 \quad \text{for all } x \in C_{r_0} \setminus C_0.$$

where :

$$\begin{aligned} KL(0, r_0) \;=\; & \{\varphi : [0, r_0] \to \mathbb{R}_+ \text{ continuous}, \\ & \varphi(0) = 0, \varphi \in C^1(0, r_0), \varphi' > 0\}. \end{aligned}$$

- Lojasiewicz inequality is a particular case with $\varphi(r) = \frac{1}{C(1-\theta)} r^{1-\theta}$.

The convex case

From now on, we assume that

$$f \text{ is } \textbf{convex}$$

Issues :

1. Characterizations of the KŁ-inequality in the convex case
2. Does a convex function satisfy the KŁ-inequality ?

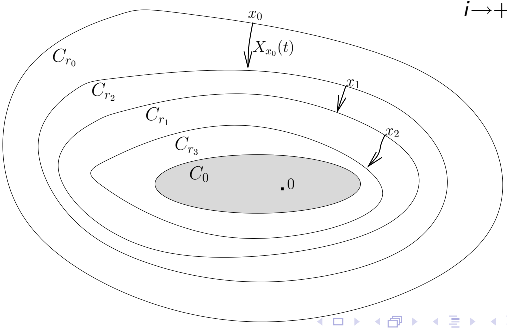
Piecewise gradient curves

Gradient dynamical system :

$$\begin{cases} \dot{X}_x(t) = -\nabla f(X_x(t)), \quad t \geq 0 \\ X_x(0) = x \end{cases}$$

A **piecewise gradient curve** γ is a countable family of gradient curves $X_{x_i}([0, t_i))$ with

$$f(X_{x_i}(0)) = f(x_i) = r_i, \qquad f(X_{x_i}(t_i)) = r_{i+1} \qquad r_i \underset{i \to +\infty}{\downarrow} 0$$



Lemma. For all $x_0 \in C_{r_0} \setminus C_0$,

1. $t \mapsto f(X_{x_0}(t))$ is convex, $L^1(0, +\infty)$ and decreasing with limit 0.
2. Each trajectory goes closer to all minima at the same time, i.e., for each $a \in C_0$,

$$\frac{d}{dt}||X_{x_0}(t) - a||^2 \leq -2f(X_{x_0}(t)) < 0.$$

3. For all $T > 0$,

$$\int_0^T ||\dot{X}_{x_0}(t)||dt \leq \frac{1}{\sqrt{2}}||x_0||\sqrt{\log T}.$$

Characterizations of the KŁ-inequality (f **convex**)

Theorem. The following statements are equivalent :

1. f satisfies the KŁ-inequality in C_{r_0} :
$||\nabla(\varphi \circ f)(x)|| \geq 1$ with $\varphi \in KL(0, r_0)$.
2. f satisfies the KŁ-inequality **globally** in H with
$\varphi \in KL(0, +\infty)$ which is **concave**.
3. $r \in (0, r_0] \mapsto \dfrac{1}{\inf\limits_{f(x)=r} ||\nabla f(x)||}$ is integrable.
4. For all piecewise gradient curves γ in C_{r_0} we have

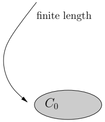
$$\mathrm{length}(\gamma) = \sum_{i=0}^{\infty} \int_0^{t_i} ||\dot{X}_{x_i}(t)||dt < \infty.$$

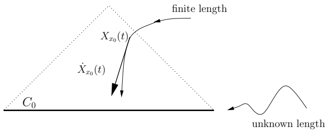
Length of the ‘convex’ gradient curves

[Brézis 1973] Given $x_0 \in C_{r_0}$, do we have

$$\mathrm{length}(X_{x_0}) = \int_0^{\infty} ||\dot{X}_{x_0}(t)||dt < \infty \ ?$$

Theorem. [Brézis 1973] Yes if $\mathrm{int}(\mathrm{argmin}(f)) \neq \emptyset$.





Theorem. [Baillon 1978] No in general (counter-example in infinite dimension)

A sufficient condition for a convex function to satisfy KŁ-inequality

Theorem. Assume that there exists $m : [0, +\infty) \to [0, +\infty)$ continuous increasing with $m(0) = 0$ such that $f \geq m(\mathrm{dist}(\cdot, C_0))$ on C_{r_0} and

$$\int_0^{r_0} \frac{m^{-1}(r)}{r} dr < +\infty \quad \text{(growth condition).} \qquad (1)$$

Then KŁ-inequality holds for f.

- Proof : $f(x) \leq \langle \nabla f(x), x - p_{C_0}(x) \rangle$
  $\leq ||\nabla f(x)|| \mathrm{dist}(x, C_0) \leq ||\nabla f(x)|| m^{-1}(f(x)).$
- non analytic examples : $m(r) = \exp(-1/r^\alpha), \alpha \in (0, 1)$ satisfies (1).

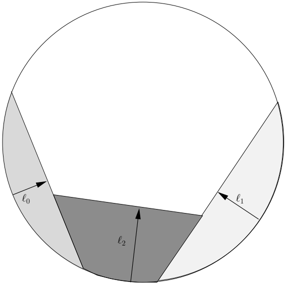
A smooth convex counterexample to KŁ-inequality

Theorem. There exists a C^2 convex function $f : \mathbb{R}^2 \to \mathbb{R}_+$ with $\{f = 0\} = D(0,1)$ for which KŁ-inequality fails.

- Note that the gradient trajectories have uniform finite length.

An auxiliary problem

A farmer rakes its (convex) field in several steps in the following way :



If he is unlucky, is it possible that he walks an infinite path ?
(i.e. $\sum_{i \geq 0} \ell_i = +\infty$)

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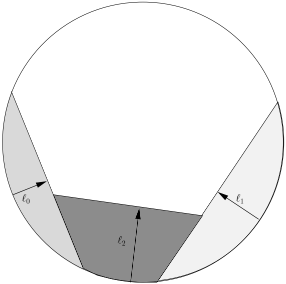

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Answer : Yes !

Lemma. There exists a decreasing sequence of compact convex subsets $\{T_k\}_k$ in $\mathbb{R}^2$ such that :

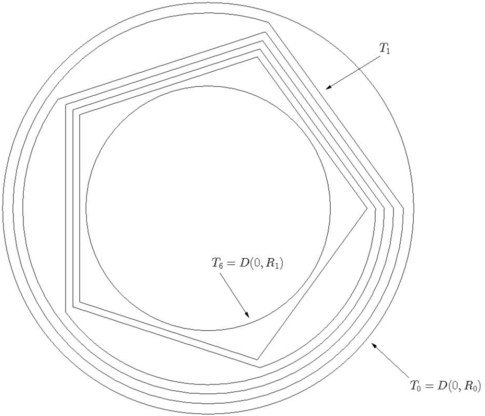
(i) T_0 is the disk $D := D(0,2)$;

(ii) $T_{k+1} \subset \mathrm{int}\; T_k$ for every $k \in \mathbb{N}$;

(iii) $\bigcap_{k\in\mathbb{N}} T_k$ is the unit disk $D(0,1)$;

(iv) $$\sum_{k=0}^{+\infty} \mathrm{dist}_{Hausdorff}(T_k, T_{k+1}) = +\infty.$$

Picture of the sequence of convex sets



$\ell_k := \mathrm{dist}_{Hausdorff}(T_k, T_{k+1}) \approx R_i - R_{i+1}$ and $N_i \approx \frac{1}{\sqrt{R_i - R_{i+1}}}$

It suffices to take $R_i - R_{i+1} = \frac{1}{i^2}$ in order that
$\sum_i R_i - R_{i+1} < \infty$ and
$\sum_k \ell_k \approx \sum_i N_i(R_i - R_{i+1}) \approx \sum_i \sqrt{R_i - R_{i+1}} = +\infty$.

End of the construction

- Construction of a convex function with prescribed sublevel sets T_k : [Torralba 1996].

- Smoothing of the function.