Efficiency for continuous facility location problems with attraction and repulsion*

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- $(X, \|\cdot\|)$ normed linear space
- $a_i^+ \in X$, $\omega_i^+ > 0 \, \forall i = 1, \cdots, m$
- $a_i^- \in X$, $\omega_i^- > 0 \, \forall i = 1, \cdots, n$

$$\boxed{(P^+) \ \min_x \sum \omega_i^+ \|x - a_i^+\|}$$

$$\boxed{(P^{+-}) \min_x \sum \omega_i^+ \|x - a_i^+\| - \sum \omega_i^- \|x - a_i^-\|}$$

M_ω set of solutions to (P^+)

- X inner product space $\Rightarrow M_\omega \subset \mathrm{co}\{a_i^+ : i = 1, \cdots, p\}$
- $\dim X = 2 \Longrightarrow M_\omega \cap \mathrm{co}\{a_i^+ : i = 1, \cdots, p\} \neq \emptyset$

Can we replace $\mathrm{co}\{a_i^+ : i = 1, \cdots, p\}$?

What happens in the case of (P^{+-}) ?

Vector Optimization

$F : Y \mapsto \mathbb{R}^p$ mapping, $D \subset Y$

$$(P) \min_{y \in D} F(y)$$

- y_0 weakly efficient if

$$\nexists y \in D; \quad F(y) - F(y_0) \in \mathrm{int}\mathbb{R}^p_-$$

- y_0 strictly efficient if

$$\nexists y \in D, y \neq y_0; \quad F(y) - F(y_0) \in \mathbb{R}^p_-$$

- y_0 efficient if

$$\nexists y \in D; \quad F(y) - F(y_0) \in \mathbb{R}^p_-, \quad F(y) \neq F(y_0)$$

Basic elements

- X euclidean
- $\Omega \subset X$ convex and closed
- $A^+ \subset X$, $A^- \subset X$ compact, $A^+ \cap A^- = \emptyset$
- $x \in X$

Weak efficiency : $x \in WE(A^+, A^-, \Omega)$

$$\forall y \in \Omega, \begin{cases} \exists a^+ \in A^+, \ \|a^+ - x\| \leq \|a^+ - y\| \\ \text{ou} \\ \exists a^- \in A^-, \ \|a^- - x\| \geq \|a^- - y\| \end{cases}$$

Strict efficiency : $x \in SE(A^+, A^-, \Omega)$

$$\forall y \in \Omega, \ y \neq x, \begin{cases} \exists a^+ \in A^+, \ \|a^+ - x\| < \|a^+ - y\| \\ \text{ou } \exists a^- \in A^-, \ \|a^- - x\| > \|a^- - y\| \end{cases}$$

Efficiency : $x \in E(A^+, A^-, \Omega)$

$$\forall y \in \Omega, \, y \neq x, \begin{cases} \exists a^+ \in A^+, \ \|a^+ - x\| < \|a^+ - y\| \\ \text{ou} \\ \exists a^- \in A^-, \ \|a^- - x\| > \|a^- - y\| \\ \text{ou} \\ \begin{cases} \forall a^+ \in A^+, \ \|a^+ - x\| \leq \|a^+ - y\| \\ \text{et} \\ \forall a^- \in A^-, \ \|a^- - x\| \geq \|a^- - y\| \end{cases} \end{cases}$$

NSOC

▶ (Carrizosa-Plastria) $A^- = \emptyset$.

x weakly efficient $\iff$

$$0 \in \mathrm{co}[\bigcup_{a^+ \in A^+} \partial(\| \cdot - a^+\|)(x)] + N(\Omega, x)$$

$A^+ = \{a_1^+, \cdots, a_n^+\}$, $A^- = \{a_1^-, \cdots, a_m^-\}$

▶ If x is locally weakly efficient, then

$$\exists \lambda^+ \geq 0,\ \exists \lambda^- \geq 0,\ \sum_{i=1}^{n} \lambda_i^+ + \sum_{j=1}^{m} \lambda_j^- = 1$$

$$[\sum_{j=1}^m \lambda_j^- \partial(\| \cdot - a_j^-\|)(x)] \bigcap [\sum_{i=1}^n \lambda_i^+ \partial(\| \cdot - a_i^+\|)(x) + N(\Omega, x)] \neq \emptyset$$

The last one is also sufficient provided that the norm and Ω are locally polyhedral.

Notations.

- $\underline{\Omega = X}$: $SE(A^+, A^-)$, $E(A^+, A^-)$, $WE(A^+, A^-)$
- $SE(A^+) = SE(A^+, \emptyset)$
- $E(A^+) = E(A^+, \emptyset)$
- $WE(A^+) = WE(A^+, \emptyset)$

Properties

▶ $SE(A^+) \subset SE(A^+, A^-)$, $E(A^+) \subset E(A^+, A^-)$, $WE(A^+) \subset WE(A^+, A^-)$

▶ $A^+ \cap \Omega \subset SE(A^+, A^-, \Omega) \subset E(A^+, A^-, \Omega) \subset WE(A^+, A^-, \Omega)$

▶ $SE(A^+, A^-) \cap \Omega \subset SE(A^+, A^-, \Omega)$

$E(A^+, A^-) \cap \Omega \subset E(A^+, A^-, \Omega)$

$WE(A^+, A^-) \cap \Omega \subset WE(A^+, A^-, \Omega)$

Weak efficiency

Theorem 1.

$$x \in WE(A^+, A^-) \Longleftrightarrow \mathrm{co}(A^+) \cap \mathrm{co}(\{x\} \cup A^-) \neq \emptyset.$$

Corollary 1.

$$\begin{array}{ccc} WE(A^+, A^-) \text{ compact} & \Longleftrightarrow & A^- = \emptyset \\ & \Updownarrow & \\ & WE(A^+, A^-) = \mathrm{co}(A^+) & \end{array}$$

Proposition 1.

$$\mathrm{co}(A^+) \cap \mathrm{co}(A^-) \neq \emptyset \Longleftrightarrow WE(A^+, A^-) = X$$

Strict efficiency

Theorem 2.

- $\mathrm{co}(A^+) \cap \mathrm{ri}[\mathrm{co}(\{x\} \cup A^-)] \neq \emptyset \Longrightarrow x \in SE(A^+, A^-)$

- $SE(A^+, A^-) = \mathrm{co}(A^+) + \mathrm{cl}[\mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]]$

Corollary 2.

- $SE(A^+, A^-)$ closed and convex

- $\mathrm{co}(A^+)$ et $\mathrm{co}(A^-)$ polyhedral :

$$SE(A^+, A^-) = \mathrm{co}(A^+) + \mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]$$

Corollary 3.

- $\mathrm{co}(A^+) \cap \mathrm{co}(A^-) \neq \emptyset \Longrightarrow \mathrm{co}(A^-) \subset SE(A^+, A^-)$

Theorem 2.

$$x \in \mathrm{ri}[SE(A^+, A^-)] \Leftrightarrow \mathrm{ri}[\mathrm{co}(A^+)] \cap \mathrm{ri}[\mathrm{co}(\{x\} \cup A^-)] \neq \emptyset.$$

Efficiency

Proposition 2.

$\mathrm{ri}[\mathrm{co}(A^+)] \cap \mathrm{ri}[\mathrm{co}(A^-)] \neq \emptyset \Longleftrightarrow E(A^+, A^-) = X$

Theorem 3.

- A^+ et A^- not contained in the same hyperplan :

$$E(A^+, A^-) = SE(A^+, A^-)$$

- A^+ et A^- contained in the same hyperplan H :

$$\mathrm{ri}[\mathrm{co}(A^+)] \cap \mathrm{ri}[\mathrm{co}(A^-)] = \emptyset \Longleftrightarrow E(A^+, A^-) \subset H$$

- $$\mathrm{ri}[\mathrm{co}(A^+)] \cap \mathrm{ri}[\mathrm{co}(A^-)] = \emptyset$$

$$\Updownarrow$$

$$E(A^+, A^-) = SE(A^+, A^-) \neq X$$

Coincidence

Theorem 5.

$K = \mathrm{co}(A^+) + \mathrm{cl}[\mathrm{cone}[\mathrm{co}(A^+) - \mathrm{co}(A^-)]]$

- $$E(A^+, A^-) = WE(A^+, A^-)$$
  ou
  $$E(A^+, A^-) = SE(A^+, A^-)$$

- $$\mathrm{co}(A^+) \cap \mathrm{co}(A^-) = \emptyset$$
  $$\Downarrow$$
  $$SE(A^+, A^-) = E(A^+, A^-) = WE(A^+, A^-) = K$$

Corollary 4.

$E(A^+, A^-)$ et $WE(A^+, A^-)$ closed and convex

Constrained efficiency

$$\mathrm{Proj}_{\Omega} WE(A^+, A^-) \subset WE(A^+, A^-, \Omega)$$

Inner product spaces

$(X, \|\cdot\|)$ linear normed space

Theorem 6. $\dim X \geq 3$.

$i)$ X inner product space

$\Updownarrow$

$ii)$ $\forall A^+, A^- \subset X$ with $A^+ \cap A^- = \emptyset$, $\mathrm{card} A^+ < +\infty$, $\mathrm{card} A^- < +\infty$, we have

$$x \in WE(A^+, A^-) \iff \mathrm{co}(A^+) \cap \mathrm{co}(\{x\} \cup A^-) \neq \emptyset.$$

Complexity

Theorem 7.

$$A^+, A^- \subset \mathbb{R}^2,\ |A^+| = n,\ |A^-| = m,\ \text{co}A^+ \cap \text{co}A^- = \emptyset$$

$$\Downarrow$$

$$SE(A^+, A^-) \text{ can be computed in } O(nm) + O(n \log n) \text{ time}$$