Optimum and equilibrium in a transport problem with queue penalization effect.

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Presentation of the Problem

- City: $\Omega \subset R^d$ bounded open set,
- k post-offices : $x_1, \ ..., x_k \in \Omega$ fixed,
- Population density: $f\ dx$ a probability,
- Unknown: partition $(A_i)_{i=1,...k}$ of Ω:
  every person linving on A_i goes to x_i.

Time lost by a citizen living at $x \in A_i$:
number of persons going to x_i: $c_i = \int_{A_i} f(x)\, dx$

$$\begin{array}{rcccl} \text{Time} & = & \text{Displacement} & + & \text{queue} \\ & = & |x - x_i|^p & + & h_i(c_i) \end{array}$$

$$p \geq 1.$$

$$\text{Total cost} = \sum_{i=1}^{k} \int_{A_i} \left(|x - x_i|^p + h_i(c_i)\right)\ f(x)dx.$$

Associated optimization problem

$$\inf_{(A_i)_i}\left\{ \sum_{i=1}^k \int_{A_i} |x-x_i|^p f(x)dx + \left(\int_{A_i} f(x)dx\right) h_i\left(\int_{A_i} f(x)dx\right) \right.$$

$$\left. (A_i)_{i=1,\ldots k} \text{ partition of } \Omega \right\}$$

$$= \inf_{(c_i)_i}\left\{ \inf_{\substack{(A_i)_i \\ \text{partition of } \Omega}} \left\{ \sum_{i=1}^k \int_{A_i} |x-x_i|^p f(x)dx : \right.\right.$$

$$\left. \int_{A_i} f(x)dx = c_i,\ c_i \geq 0 \right\}$$

$$\left. + \sum_i c_i h_i(c_i) :\ \sum_i c_i = 1 \right\}$$

$$= \inf_{(c_i)_i}\left\{ w_p^p(fdx, \textstyle\sum_{i=1}^k c_i\delta_{x_i}) + \sum_{i=1}^k c_i h_i(c_i) :\ c_i \geq 0,\ \sum_i c_i = 1 \right\}$$

$W_p(fdx, \sum_{i=1}^k c_i\delta_{x_i})$ is the p-Wasserstein distance.

Wasserstein distance from fdx to $\sum_{i=1}^k c_i\delta_{x_i}$

$$W_p(fdx, \sum_{i=1}^{k} c_i\delta_{x_i}) = \inf_T \int_\Omega |x - Tx|^p f(x)dx$$

$T(x) = x_i \ \forall x \in A_i$ where $(A_i)_{i=1,\ldots k}$ is a partition Ω such that: $\int_{A_i} f(x)dx = c_i$.

Existence of a transport map

It exists an optimal transport map T for W_p.

- $p = 2$ **Brenier** (87).
- $p > 1$ **Rüschendorf** (95), **Gangbo, McCann** (96).
- $p = 1$ **Sudakov** (79), **Gangbo, McCann** (96), **Evans, Gangbo** (99), **Caffarelli, Feldmann, McCann** (02), **Ambrosio, Pratelli** (03)...

Optimality transportation

Kantorovich duality

$$W_p(fdx, \textstyle\sum_{i=1}^k c_i\delta_{x_i})$$

$$= \sup_{u,\ (\alpha_i)_i} \left\{ \int_\Omega u(x)\, f(x)dx + \sum_{i=1}^k c_i\alpha_i : \quad u \in \mathrm{L}^1(\Omega),\right.$$

$$\left. u(x) - \alpha_i \leq |x - x_i|^p \quad \text{a.e.} x \in \Omega \right\}$$

Primal-Dual Optimality conditions (Bouchitté)

$$\left.\begin{array}{c}(u,(\alpha_i)_i,T)\\ \text{optimal}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} T=\sum_i x_i 1_{A_i}\\ u(x)=\inf_i |x-x_i|^p-\alpha_i\\ \qquad\quad =\sum_i \left(|x-x_i|^p-\alpha_i\right)1_{A_i}\\ A_i=\{x\in\Omega:\ |x-x_i|^p-\alpha_i<|x-x_j|^p-\alpha_j\}\end{array}\right.$$

$$\left.\begin{array}{r} u(x)=\inf_i |x-x_i|^p-\alpha_i\\ =\sum_i \left(|x-x_i|^p-\alpha_i\right)1_{A_i}\end{array}\right\} \Rightarrow \left\{\begin{array}{l} T=\sum_i x_i 1_{A_i} \mathit{and}\\ u,(\alpha_i)_i \text{ optimal.}\end{array}\right.$$

Consequence: the optimal partition is always unique.

The optimization problem

$$\inf_{(c_i)_i}\{W_p^p(fdx, \textstyle\sum_{i=1}^k c_i\delta_{x_i}) + \sum_{i=1}^k c_i h_i(c_i) : \; c_i \geq 0, \; \sum_i c_i = 1\}$$

$$W_p^p(fdx, \sum_{i=1}^k c_i\delta_{x_i}) = \inf_{(A_i)_i}\{\sum_i \int_{A_i} |x - x_i| \, f(x)dx\}$$

under the constraints: $(A_i)_i$ is a partition of Ω, $c_i = \int_{A_i} f(x)dx$.

Existence of an optimum

If $t \mapsto th_i(t)$ is l.s.c. for any $i = 1, ..k$, it exists an optimal partition of Ω. Moreover if $t \mapsto th_i(t)$ is strictly convex, the optimal partition is unique.

The optimization problem

Necessary and Sufficient Optimality Condition

We assume h_i is regular and $t \mapsto th_i(t)$ is convex for any $i = 1, ..k$, then a partition $(A_i)_i$ is optimal iff (up to negligible sets):

$$\begin{cases} A_i = \{x \in \Omega : & |x - x_i|^p + h_i(c_i) + c_i h_i'(c_i) \\ & < |x - x_j|^p + h_j(c_j) + c_j h_j'(c_j)\} \\ c_i = \int_{A_i} f(x)dx. & \end{cases}$$

Moreover, there is only one partition which satisfies this condition.

Example

Let $\Omega = [0,1]$, $f = 1$, $x_1 = 0$, $x_2 = 1$, $p = 1$ and:

$$h_1(s) = 100, \quad \text{and } h_2(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq 0.999 \\ 1 & \text{for } 0.999 < s \leq 1. \end{cases}$$

Optimum: $A_1 = [0, 0.001[$, $A_2 =]0.001, 1]$...
Costumers in A_1 may not be happy!
A costumer living at $x \in A_i$ will be happy if:

$$|x - x_i|^p + h_i(c_i) = \inf_j \{|x - x_j|^p + h_i(c_j)\}.$$

In the example:

$$A_1 = \emptyset, \; A_2 = [0,1].$$

The equilibrium problem

Nash Equilibrium

A partition $(A_i)_{i=1,..k}$ is a Nash equilibrium if:

$$\begin{cases} A_i = \{x \in \Omega : \ |x - x_i|^p + h_i(c_i) < |x - x_j|^p + h_j(c_j)\} \\ c_i = \int_{A_i} f(x)dx. \end{cases}$$

Existence

Assume h_i is continuous for any $i = 1, ..k$ and g_i is such that $tg_i'(t) + g_i(t) = h_i(t)$.
Then, the minimizer of the following problem is an equilibrium:

$$\inf_{\substack{(A_i)_i \\ \text{partition of}\Omega}} \left\{ \sum_i \int_{A_i} |x - x_i|^p + g_i\left(\int_{A_i} f(x)dx\right) f(x)dx \right\}.$$

If, in addition, h_i is non-decreasing, the equilibrium is unique.

Example

Travellers game

	Traveller 1 $2 \leq t_1 \leq 100$ $t_1 \in N$	Traveller 2 $2 \leq t_2 \leq 100$ $t_2 \in N$
$t_1 = t_2$	t_1	t_2
$t_1 < t_2$	$t_1 + 2$	$t_1 - 2$
$t_1 > t_2$	$t_2 - 2$	$t_2 + 2$

Equilibrium: $t_1 = t_2 = 2$.

Pareto Optimum

If $(A_i)_i$ is an equilibrium, is it possible to find another partition $(B_i)_i$ such that every citizen spends equal or less time ?
Individual cost:

$$C(x,(B_i)_i) = \sum_{i=1}^{k} \left(|x - x_i|^p + h_i \left(\int_{B_i} f(x)dx \right) \right) 1_{B_i}(x).$$

Pareto Optimum

A partition $(A_i)_{i=1,..k}$ is a Pareto optimum if there exists **NO** other partition $(B_i)_i$ of Ω such that:

$$C(x,(B_i)_i) < C(x,(A_i)_i) \quad \text{a.e. } x \in \Omega.$$

Proposition

Assume h_i is strictly increasing. Then every equilibrium is a Pareto optimum.

Sketch of proof

Important remark

(A_i) is an equilibrium $\Leftrightarrow$
$C(x,(A_i)_i) = \inf_{i=1,\dots k} \left(|x - x_i|^p + h_i \left(\int_{A_i} f(x)dx \right) \right)$.

Remember the optimality condition for optimal transportation (slide 5)

Sketch of proof

Link between equilibrium and optimal transportation

- If $(A_i)_i$ is an equilibrium then:
  $T = \sum_i x_i 1_{A_i}$ is optimal for $W_p(f, \sum_i \left(\int_{A_i} f(x)dx\right) \delta_{x_i})$,
  $u(x) = C(x, (A_i)_i)$ and $(\alpha_i)_i = -h_i \left(\int_{A_i} f(x)dx\right)$ are optimal for the dual formulation of W_p.
- If $(A_i)_i$ is a partition such that $u(x) = C(x, (A_i)_i)$ and $(\alpha_i)_i = -h_i \left(\int_{A_i} f(x)dx\right)$ are optimal for the dual formulation of W_p, then:
  $(A_i)_i$ is an equilibrium
  $T = \sum_i x_i 1_{A_i}$ is optimal for $W_p(f, \sum_i \left(\int_{A_i} f(x)dx\right) \delta_{x_i})$.

Sketch of proof

Assume $h_i \nearrow$, $(A_i)_i$ is an equilibrium,
$(B_i)_i$ is such that $C(x,(B_i)_i) \leq C(x,(A_i)_i)$a.e.

- Let us show: $\int_{A_i} f(x)dx = \int_{B_i} f(x)dx$ for every i.
  Assume $\exists j$ such that $\int_{A_j} f(x)dx < \int_{B_j} f(x)dx$. Then:

$$\begin{aligned}
\forall x \in B_j \qquad & |x - x_j|^p + h_j\Big(\int_{B_j} f(x)dx\Big) \\
&= C(x,(B_i)_i) \leq C(x,(A_i)_i) \\
&= \min_i \{|x - x_i|^p + h_i\Big(\int_{A_i} f(x)dx\Big)\} \\
&\leq |x - x_j|^p + h_j\Big(\int_{A_j} f(x)dx\Big).
\end{aligned}$$

This is impossible because h_j is strictly increasing.

Sketch of proof

- As $(A_i)_i$ is an equilibrium, $T = \sum_i x_i 1_{A_i}$ is an optimal transport map for $W_p(f, \sum_i \left(\int_{A_i} f(x)dx\right) \delta_{x_i})$, by consequence:

$$\sum_i \int_{A_i} |x - x_i|^p \, f(x)dx \leq \sum_i \int_{B_i} |x - x_i|^p \, f(x)dx.$$

But as $C(x, (B_i)_i) \leq C(x, (A_i)_i)$ and $\int_{A_i} f(x)dx = \int_{B_i} f(x)dx$:

$$\sum_i \int_{A_i} |x - x_i|^p \, f(x)dx \geq \sum_i \int_{B_i} |x - x_i|^p \, f(x)dx.$$

By uniqueness of the optimal transport map, $(A_i)_i = (B_i)_i$ up to negligible sets.

Dynamic behaviour for 2 points

For simplicity, assume:

$$\Omega = [0,1],\ x_1 = 0,\ x_2 = 1,\ f = 1,\ p = 1.$$

At every day n, we set t_n such that:

$$A_1 = [0, t_n[,\ A_2 =]t_n, 1].$$

- Day 1: the citizen has no information on the queue:
  $t_1 = 1/2$,
  End of day: he knows $h_1(t_1)$, $h_2(1-t_1)$.
- Day n: he knows $h_1(t_{n-1})$, $h_2(1-t_{n-1})$,

$$t_{n+1} + h_1(t_n) = (1 - t_{n+1}) + h_2(1 - t_n)$$

$$t_{n+1} := \frac{h_2(1-t_n) - h_1(t_n) + 1}{2}.$$

- Equilibrium: $\bar{t}$ such that: $\bar{t} = \frac{h_2(1-\bar{t}) - h_1(\bar{t}) + 1}{2}$.

Dynamic behaviour for 2 points

Convergence

Let $G(t) := \frac{h_2(1-t)-h_1(t)+1}{2}$. If G is a contraction mapping, then $t_n \longrightarrow \bar{t}$.

Dynamic with memory

Assume the citizen "remembers" the K last days. The map G is the same as before but we set:

$$t_{n+1} = \frac{\sum_{i=1}^{K} G(t_{n-K+i})}{K}.$$

Convergence

Assume G is L-lipschitz with $L < K$, then $t_n \to t$.

Sketch of proof for $K = 2$

Assume that for every $k \leq n$:

$$\begin{cases} (i) & |G(t_k) - G(\bar{t})| \leq \alpha \\ (ii) & |G(t_k) - G(\bar{t}) + G(t_{k-1}) - G(\bar{t})| \leq \alpha \end{cases}$$

- Recall $t_{n+1} = \frac{G(t_n)+G(t_{n-1})}{2}$.

$$\begin{aligned} |G(t_{n+1}) - G(\bar{t})| &\leq L|t_{n+1} - \bar{t}| \quad \text{Lipschitz Property} \\ &= L|t_{n+1} - G(\bar{t})| \quad \bar{t} \text{ is a fixed point} \\ &= L\left|\frac{G(t_n) - G(\bar{t}) + G(t_{n-1}) - G(\bar{t})}{2}\right| \\ &\leq \frac{L\alpha}{2} \quad \text{by (ii)} . \end{aligned}$$

Sketch of proof for $K = 2$

- Note that $G \searrow$.

(case 1) If $(G(t_{n+1}) - G(\bar{t})$ and $(G(t_n) - G(\bar{t})$ have different signs we have easily:

$$|G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})| \leq \alpha.$$

(case 2) Let us assume $t_n,\ t_{n+1} > \bar{t}$ so that $G(t_n) - G(\bar{t}) \leq 0$. Then:

$$t_{n+1} - \bar{t} = t_{n+1} = \frac{G(t_n) + G(t_{n-1})}{2} - \bar{t} > 0 \Rightarrow G(t_{n-1}) - G(\bar{t}) \geq 0.$$

Hence:

$$\begin{aligned} & |G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})| \\ = \ & |G(\bar{t}) - G(t_{n+1}) + G(\bar{t}) - G(t_n)| \\ \leq \ & L(t_{n+1} - G(\bar{t}) + G(\bar{t}) - G(t_n)) \\ = \ & L\left(\frac{G(t_n) + G(t_{n-1})}{2} - G(\bar{t})\right) + G(\bar{t}) - G(t_n) \\ \leq \ & (1 - L/2)\alpha + L/2\alpha = \alpha. \end{aligned}$$

Sketch of proof for $K = 2$

At time $n + 1$ we have:

$$\begin{cases} (i) & |G(t_{n+1}) - G(\bar{t})| \leq \frac{L\alpha}{2} \\ (ii) & |G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})| \leq \alpha \end{cases}$$

By repeating exactly the same arguments:

$$\begin{cases} (i) & |G(t_{n+2}) - G(\bar{t})| \leq \frac{L\alpha}{2} \\ (ii) & |G(t_{n+2}) - G(\bar{t}) + G(t_{n+1}) - G(\bar{t})| \leq \frac{L\alpha}{2} \end{cases}$$

etc...

Dynamic with memory

Convergence

Assume G is strictly L-lipschitz with $L \leq K$, then $t_n \to t$.

Infinite memory:

$$t_n = \frac{\sum_{i=1}^{n-1} G(t_i)}{n-1}.$$

Convergence

Assume G is lipschitz, then $t_n \to t$.