Optimum and equilibrium in a transport problem with queue penalization effect.

G. Crippa,
(University of Parma (Italy))

C. Jimenez,
(Université de Bretagne Occidentale)

A. Pratelli,
(University of Pavia (Italy).)
Presentation of the Problem

- City: $\Omega \subset R^d$ bounded open set,
- $k$ post-offices: $x_1, \ldots, x_k \in \Omega$ fixed,
- Population density: $f \, dx$ a probability,
- Unknown: partition $(A_i)_{i=1}^k$ of $\Omega$:
  every person living on $A_i$ goes to $x_i$.

**Time lost by a citizen living at** $x \in A_i$:
**number of persons going to** $x_i$: $c_i = \int_{A_i} f(x) \, dx$

\[
\text{Time} = \text{Displacement} + \text{queue} = |x - x_i|^p + h_i(c_i)
\]

$p \geq 1$.

Total cost = $\sum_{i=1}^{k} \int_{A_i} (|x - x_i|^p + h_i(c_i)) \, f(x) \, dx$. 

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Associated optimization problem

\[ \inf_{(A_i)_i} \left\{ \sum_{i=1}^{k} \int_{A_i} |x - x_i|^p f(x) \, dx + \left( \int_{A_i} f(x) \, dx \right) h_i \left( \int_{A_i} f(x) \, dx \right) \right\} \]

\( (A_i)_{i=1,...,k} \text{ partition of } \Omega \}

\[ = \inf_{(c_i)_i} \left\{ \inf_{(A_i)_i} \left\{ \sum_{i=1}^{k} \int_{A_i} |x - x_i|^p f(x) \, dx : \text{partition of } \Omega \right\} \int_{A_i} f(x) \, dx = c_i, \; c_i \geq 0 \right\} \]

\[ + \sum_i c_i h_i(c_i) : \sum_i c_i = 1 \}

\[ = \inf_{(c_i)_i} \left\{ w_p^p(fdx, \sum_{i=1}^{k} c_i \delta_{x_i}) + \sum_{i=1}^{k} c_i h_i(c_i) : c_i \geq 0, \sum_i c_i = 1 \right\} \]

\( W_p(fdx, \sum_{i=1}^{k} c_i \delta_{x_i}) \) is the \( p \)-Wasserstein distance.
Optimal Transportation

Wasserstein distance from $f dx$ to $\sum_{i=1}^{k} c_i \delta_{x_i}$

$$W_p(f dx, \sum_{i=1}^{k} c_i \delta_{x_i}) = \inf_{T} \int_{\Omega} |x - Tx|^p f(x) dx$$

$$T(x) = x_i \ \forall x \in A_i \text{ where } (A_i)_{i=1}^k \text{ is a partition } \Omega \text{ such that:}$$

$$\int_{A_i} f(x) dx = c_i.$$ 

Existence of a transport map

It exists an optimal transport map $T$ for $W_p$.

- $p = 2$ Brenier (87).
- $p > 1$ Rüschendorf (95), Gangbo, McCann (96).
- $p = 1$ Sudakov (79), Gangbo, McCann (96), Evans, Gangbo (99), Caffarelli, Feldmann, McCann (02), Ambrosio, Pratelli (03)...
Kantorovich duality

\[ W_p(fdx, \sum_{i=1}^k c_i \delta_{x_i}) = \sup_{u, (\alpha_i)_i} \left\{ \int_{\Omega} u(x) f(x) dx + \sum_{i=1}^k c_i \alpha_i : \ u \in L^1(\Omega), \right. \]

\[ u(x) - \alpha_i \leq |x - x_i|^p \quad \text{a.e.} \ x \in \Omega \]
Primal-Dual Optimality conditions (Bouchitté)

\[(u, (\alpha_i)_i, T) \text{ optimal} \implies \left\{ \begin{array}{l}
T = \sum_i x_i 1_{A_i} \\
u(x) = \inf_i |x - x_i|^p - \alpha_i = \sum_i (|x - x_i|^p - \alpha_i) 1_{A_i} \\
A_i = \{x \in \Omega : |x - x_i|^p - \alpha_i < |x - x_i|^p - \alpha_j\}
\end{array} \right.\]

Consequence: the optimal partition is always unique.
The optimization problem

\[ \inf_{(c_i)_i} \{ W_p^p(fdx, \sum_{i=1}^k c_i \delta_{x_i}) + \sum_{i=1}^k c_i h_i(c_i) : c_i \geq 0, \sum_i c_i = 1 \} \]

\[ W_p^p(fdx, \sum_{i=1}^k c_i \delta_{x_i}) = \inf_{(A_i)_i} \left\{ \sum_i \int_{A_i} |x - x_i| f(x) dx \right\} \]

under the constraints: \((A_i)_i\) is a partition of \(\Omega\), \(c_i = \int_{A_i} f(x) dx\).

Existence of an optimum

If \(t \mapsto th_i(t)\) is l.s.c. for any \(i = 1, \ldots, k\), it exists an optimal partition of \(\Omega\). Moreover if \(t \mapsto th_i(t)\) is strictly convex, the optimal partition is unique.

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The optimization problem

Necessary and Sufficient Optimality Condition

We assume $h_i$ is regular and $t \mapsto th_i(t)$ is convex for any $i = 1, \ldots, k$, then a partition $(A_i)_i$ is optimal iff (up to negligible sets):

$$
A_i = \{ x \in \Omega : |x - x_i|^p + h_i(c_i) + c_i h'(c_i) < |x - x_j|^p + h_j(c_j) + c_j h'(c_j) \} 
$$

$$
c_i = \int_{A_i} f(x) \, dx.
$$

Moreover, there is only one partition which satisfies this condition.
Let $\Omega = [0, 1]$, $f = 1$, $x_1 = 0$, $x_2 = 1$, $p = 1$ and:

$$h_1(s) = 100, \quad \text{and} \quad h_2(s) = \begin{cases} 0 & \text{for } 0 \leq s \leq 0.999 \\ 1 & \text{for } 0.999 < s \leq 1. \end{cases}$$

Optimum: $A_1 = [0, 0.001[$, $A_2 = ]0.001, 1]\ldots$

Costumers in $A_1$ may not be happy!

A costumer living at $x \in A_i$ will be happy if:

$$|x - x_i|^p + h_i(c_i) = \inf_j \{|x - x_j|^p + h_i(c_j)\}.$$ 

In the example:

$$A_1 = \emptyset, \quad A_2 = [0, 1].$$
The equilibrium problem

Nash Equilibrium

A partition \((A_i)_{i=1,..,k}\) is a Nash equilibrium if:

\[
\begin{aligned}
A_i &= \{x \in \Omega : |x - x_i|^p + h_i(c_i) < |x - x_j|^p + h_j(c_j)\} \\
c_i &= \int_{A_i} f(x) dx.
\end{aligned}
\]

Existence

Assume \(h_i\) is continuous for any \(i = 1,..k\) and \(g_i\) is such that

\[tg'_i(t) + g_i(t) = h_i(t).\]

Then, the minimizer of the following problem is an equilibrium:

\[
\inf_{(A_i)_i} \left\{ \sum_i \int_{A_i} |x - x_i|^p + g_i \left( \int_{A_i} f(x) dx \right) f(x) dx \right\}.
\]

If, in addition, \(h_i\) is non-decreasing, the equilibrium is unique.
Example

Travellers game

<table>
<thead>
<tr>
<th></th>
<th>Traveller 1</th>
<th>Traveller 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2 \leq t_1 \leq 100$</td>
<td>$2 \leq t_2 \leq 100$</td>
</tr>
<tr>
<td></td>
<td>$t_1 \in \mathbb{N}$</td>
<td>$t_2 \in \mathbb{N}$</td>
</tr>
<tr>
<td>$t_1 = t_2$</td>
<td>$t_1$</td>
<td>$t_2$</td>
</tr>
<tr>
<td>$t_1 &lt; t_2$</td>
<td>$t_1 + 2$</td>
<td>$t_2 - 2$</td>
</tr>
<tr>
<td>$t_1 &gt; t_2$</td>
<td>$t_2 - 2$</td>
<td>$t_2 + 2$</td>
</tr>
</tbody>
</table>

Equilibrium: $t_1 = t_2 = 2$. 
If \((A_i)_i\) is an equilibrium, is it possible to find another partition \((B_i)_i\) such that every citizen spends equal or less time?

**Individual cost:**

\[
C(x, (B_i)_i) = \sum_{i=1}^{k} \left( |x - x_i|^p + h_i \left( \int_{B_i} f(x) dx \right) \right) 1_{B_i}(x).
\]

**Pareto Optimum**

A partition \((A_i)_{i=1,..,k}\) is a Pareto optimum if there exists NO other partition \((B_i)_i\) of \(\Omega\) such that:

\[
C(x, (B_i)_i) < C(x, (A_i)_i) \quad \text{a.e.} \ x \in \Omega.
\]

**Proposition**

Assume \(h_i\) is strictly increasing. Then every equilibrium is a Pareto optimum.
Important remark

$(A_i)$ is an equilibrium $\iff$

\[ C(x, (A_i)_i) = \inf_{i=1,\ldots,k} \left( |x - x_i|^p + h_i \left( \int_{A_i} f(x) \, dx \right) \right). \]

Remember the optimality condition for optimal transportation (slide 5)
If \((A_i)_i\) is an equilibrium then:
\[
T = \sum_i x_i 1_{A_i} \text{ is optimal for } W_p(f, \sum_i \left( \int_{A_i} f(x)dx \right) \delta_{x_i}),
\]
\[
u(x) = C(x, (A_i)_i) \text{ and } (\alpha_i)_i = -h_i \left( \int_{A_i} f(x)dx \right) \text{ are optimal for the dual formulation of } W_p.
\]
If \((A_i)_i\) is a partition such that \(u(x) = C(x, (A_i)_i)\) and \((\alpha_i)_i = -h_i \left( \int_{A_i} f(x)dx \right)\) are optimal for the dual formulation of \(W_p\), then:

\((A_i)_i\) is an equilibrium
\[
T = \sum_i x_i 1_{A_i} \text{ is optimal for } W_p(f, \sum_i \left( \int_{A_i} f(x)dx \right) \delta_{x_i}).
\]
Assume $h_i \nearrow$, $(A_i)_i$ is an equilibrium, $(B_i)_i$ is such that $C(x, (B_i)_i) \leq C(x, (A_i)_i)$ a.e.

Let us show: $\int_{A_i} f(x)dx = \int_{B_i} f(x)dx$ for every $i$.

Assume $\exists j$ such that $\int_{A_j} f(x)dx < \int_{B_j} f(x)dx$. Then:

$$\forall x \in B_j \quad |x - x_j|^p + h_j(\int_{B_j} f(x)dx)$$

$$= C(x, (B_i)_i) \leq C(x, (A_i)_i)$$

$$= \min_i \{|x - x_i|^p + h_i(\int_{A_i} f(x)dx)\}$$

$$\leq |x - x_j|^p + h_j(\int_{A_j} f(x)dx).$$

This is impossible because $h_j$ is strictly increasing.
As \((A_i)_i\) is an equilibrium, \(T = \sum_i x_i 1_{A_i}\) is an optimal transport map for \(W_p(f, \sum_i \left(\int_{A_i} f(x) dx\right) \delta_{x_i})\), by consequence:

\[
\sum_i \int_{A_i} |x - x_i|^p f(x) dx \leq \sum_i \int_{B_i} |x - x_i|^p f(x) dx.
\]

But as \(C(x, (B_i)_i) \leq C(x, (A_i)_i)\) and \(\int_{A_i} f(x) dx = \int_{B_i} f(x) dx\):

\[
\sum_i \int_{A_i} |x - x_i|^p f(x) dx \geq \sum_i \int_{B_i} |x - x_i|^p f(x) dx.
\]

By uniqueness of the optimal transport map, \((A_i)_i = (B_i)_i\) up to negligible sets.
Dynamic behaviour for 2 points

For simplicity, assume:

\[ \Omega = [0, 1], \ x_1 = 0, \ x_2 = 1, \ f = 1, \ p = 1. \]

At every day \( n \), we set \( t_n \) such that:

\[ A_1 = [0, t_n[, \ A_2 = ]t_n, 1]. \]

- Day 1: the citizen has no information on the queue:
  \( t_1 = 1/2 \),
  End of day: he knows \( h_1(t_1), h_2(1 - t_1) \).
- Day \( n \): he knows \( h_1(t_{n-1}), h_2(1 - t_{n-1}) \),

\[ t_{n+1} + h_1(t_n) = (1 - t_{n+1}) + h_2(1 - t_n) \]

\[ t_{n+1} := \frac{h_2(1 - t_n) - h_1(t_n) + 1}{2}. \]

- Equilibrium: \( \bar{t} \) such that:

\[ \bar{t} = \frac{h_2(1 - \bar{t}) - h_1(\bar{t}) + 1}{2}. \]
Dynamic behaviour for 2 points

Let $G(t) := \frac{h_2(1-t) - h_1(t) + 1}{2}$. If $G$ is a contraction mapping, then $t_n \to \bar{t}$.

Convergence
Assume the citizen "remembers" the $K$ last days. The map $G$ is the same as before but we set:

$$t_{n+1} = \frac{\sum_{i=1}^{K} G(t_{n-K+i})}{K}.$$  

**Convergence**

Assume $G$ is $L$-lipschitz with $L < K$, then $t_n \rightarrow t$. 
Assume that for every $k \leq n$:

$$
\begin{align*}
(i) & \quad |G(t_k) - G(\bar{t})| \leq \alpha \\
(ii) & \quad |G(t_k) - G(\bar{t}) + G(t_{k-1}) - G(\bar{t})| \leq \alpha
\end{align*}
$$

Recall $t_{n+1} = \frac{G(t_n) + G(t_{n-1})}{2}$.

$$
|G(t_{n+1}) - G(\bar{t})| \leq L|t_{n+1} - \bar{t}| \quad \text{Lipschitz Property}
$$

$$
= L|t_{n+1} - G(\bar{t})| \quad \bar{t} \text{ is a fixed point}
$$

$$
= L \left| \frac{G(t_n) - G(\bar{t}) + G(t_{n-1}) - G(\bar{t})}{2} \right|
$$

$$
\leq \frac{L\alpha}{2} \quad \text{by (ii)}.
$$
Sketch of proof for $K = 2$

Note that $G \downarrow$.

(case 1) If $(G(t_{n+1}) - G(\bar{t})$ and $(G(t_n) - G(\bar{t})$ have different signs we have easily:

$$|G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})| \leq \alpha.$$ 

(case 2) Let us assume $t_n, t_{n+1} > \bar{t}$ so that $G(t_n) - G(\bar{t}) \leq 0$. Then:

$$t_{n+1} - \bar{t} = t_{n+1} = \frac{G(t_n) + G(t_{n-1})}{2} - \bar{t} > 0 \Rightarrow G(t_{n-1}) - G(\bar{t}) \geq 0.$$ 

Hence:

$$|G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})|$$

$$= |G(\bar{t}) - G(t_{n+1}) + G(\bar{t}) - G(t_n)|$$

$$\leq L(t_{n+1} - G(\bar{t}) + G(\bar{t}) - G(t_n))$$

$$= L \left( \frac{G(t_n) + G(t_{n-1})}{2} - G(\bar{t}) \right) + G(\bar{t}) - G(t_n)$$

$$\leq (1 - L/2)\alpha + L/2\alpha = \alpha.$$
Sketch of proof for $K = 2$

At time $n + 1$ we have:

\[
\begin{align*}
(i) \quad |G(t_{n+1}) - G(\bar{t})| & \leq \frac{L\alpha}{2} \\
(ii) \quad |G(t_{n+1}) - G(\bar{t}) + G(t_n) - G(\bar{t})| & \leq \alpha
\end{align*}
\]

By repeating exactly the same arguments:

\[
\begin{align*}
(i) \quad |G(t_{n+2}) - G(\bar{t})| & \leq \frac{L\alpha}{2} \\
(ii) \quad |G(t_{n+2}) - G(\bar{t}) + G(t_{n+1}) - G(\bar{t})| & \leq \frac{L\alpha}{2}
\end{align*}
\]

etc...
Convergence

Assume $G$ is strictly $L$-lipschitz with $L \leq K$, then $t_n \to t$.

Infinite memory:

$$t_n = \frac{\sum_{i=1}^{n-1} G(t_i)}{n-1}.$$