Maximal Monotonicity of Bifunctions

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1 Monotone Bifunctions

X: Banach space; $C \subseteq X$ nonempty.

Bifunction: any function $F : C \times C \to \mathbb{R}$ such that $F(x,x) = 0$, $\forall x \in C$.

Equilibrium problem: Find $x_0 \in C$ such that

$$\forall y \in C, \quad F(x_0, y) \geq 0. \tag{EP}$$

Most famous paper on equilibrium problems: E. Blum and W. Oettli, *The Mathematics Student* (1994).

Example: Given a multivalued operator $T : C \to 2^{X^*} \backslash \{\emptyset\}$ with weak*-compact values, set

$$F(x, y) = \max_{x^* \in T(x)} \langle x^*, y - x \rangle .$$

Then F is a bifunction ($F(x, x) = 0$) and (EP) is equivalent to the variational inequality problem: find $x_0 \in C$ such that

$$\forall y \in C, \exists x^* \in T(x_0), \quad \langle x^*, y - x \rangle \geq 0. \qquad \text{(VIP)}$$

Other examples: Mathematical Programming problem, fixed point problem, saddle point problem...

$T : X \to 2^{X^*}$ an operator.

$D(T) = \{x \in X : T(x) \neq \emptyset\}$ the domain of T.

$\operatorname{gr} T = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ the graph of T.

T is called monotone if

$$\forall (x, x^*), (y, y^*) \in \operatorname{gr} T : \langle y^* - x^*, y - x \rangle \geq 0.$$

A monotone operator T is called maximal monotone if there is no monotone operator S with $\operatorname{gr} T \subsetneq \operatorname{gr} S$.

A bifunction F is called monotone if

$$\forall x, y \in C, \quad F(x, y) + F(y, x) \leq 0.$$

Example: If $T : X \to 2^{X^*}$ is any monotone operator, define the bifunction $G_T : D(T) \times D(T) \to \mathbb{R}$ by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle .$$

Since T is monotone, G_T is also monotone.

Starting from a monotone bifunction $F : C \times C \to \mathbb{R}$, one can define a monotone operator A^F as follows: for any $x \in C$ set

$$A^F(x) = \{x^* \in X^* : \forall y \in C, F(x,y) \geq \langle x^*, y - x\rangle\}$$

while $A^F(x) = \emptyset$ if $X \backslash C$.

If $\hat{F} : C \times X \to \mathbb{R} \cup \{+\infty\}$ is the extension of F defined by setting $\hat{F}(x,y) = +\infty$ for $x \in C$, $y \in X \backslash C$, then

$$\forall x \in C, \quad A^F(x) = \partial \hat{F}(x, \cdot)(x).$$

Special case: if $f : X \to \mathbb{R} \cup \{+\infty\}$ with **dom** $f = C$ and $F(x,y) = f(y) - f(x)$, then F is monotone and $A^F = \partial f$. In more general cases, A^F is not the subdifferential of a unique function. However, it is monotone (same proof as for the subdifferential), and $A^F(x)$ is closed and convex.

Note: Different bifunctions F_1 and F_2 on C may produce the same operator: $A^{F_1} = A^{F_2}$.

Definition 1 *A monotone bifunction F is called maximal monotone, if A^F is maximal monotone.*

Another definition of maximal monotonicity (Blum-Oettli): F is BO-maximal monotone if for every $x \in C$, $x^* \in X^*$ the following implication holds:

$$\forall y \in C, F(y,x) + \langle x^*, y - x \rangle \leq 0$$
$$\Rightarrow \forall y \in C, F(x,y) \geq \langle x^*, y - x \rangle .$$

If C is convex and $F(x,\cdot)$ is convex and l.s.c., the two definitions of maximal monotonicity are equivalent (Ait Mansour, Chbani and Riahi, 2003). This is not the case in general.

Example: $f : X \to \mathbb{R}$ any odd function, $F(x,y) = f(y-x)$. Then F is monotone, and in fact BO-maximal monotone. One can check: F is maximal monotone if and only if $f \in X^*$. In fact, if $f \notin X^*$, then $D(A^F) = \emptyset$. For instance, $F(x,y) = (y-x)^3$ is BO-maximal monotone, but not maximal monotone.

Proposition 1 *If the operator T is maximal monotone, then G_T is maximal monotone. In fact, $A^{G_T} = T$.*

The converse does not hold: If G_T is maximal monotone, then T is not necessarily maximal monotone, as it is possible to have $T \neq S$ and $G_T = G_S$. For instance, if T is maximal monotone and $S \neq T$ but $\overline{\text{co}}S(x) = T(x)$ for all $x \in X$, then $G_S = G_T$ hence G_S is maximal monotone, while S is not. Another example: T is defined on $\mathbb{R}$, $D(T) = [0, +\infty)$, $T(x) = \{0\}$ for $x \in [0, +\infty)$. Then G_T is maximal, T is not maximal.

Proposition 2 *Assume that T has closed convex values and $D(T) = X$. If G_T is maximal monotone, then T is maximal monotone.*

Starting from a maximal monotone bifunction F, one may construct A^F and then the maximal monotone bifunction $G := G_{A^F}$. In general, $G(x, y) \leq F(x, y)$ for $x, y \in D(A^F)$, $A^F = A^G$, but $F \neq G$. Example: $F(x, y) = y^2 - x^2$, then $A^F(x) = \{2x\}$, $G(x) = 2x(y - x)$.

2 Conditions for maximality of bifunctions

In the following, X will be reflexive, equipped with a norm such that both X and X^* are strictly convex ($\|x\|+\|y\| = \|x+y\|$ implies that x, y are proportional). Then for each $x \in X$ there exists a unique $x^* \in X^*$ such that $\langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2$. The correspondence $x \to \mathcal{J}(x) := x^*$ is the so-called duality map.

The following proposition is a consequence of the results of Ait Mansour/Chbani/Riahi, 2003.

Proposition 3 *A monotone bifunction F is maximal monotone if and only if for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) and each $x \in X$ there exists $x_\lambda \in C$ such that*

$$\forall y \in C, \quad \lambda F(x_\lambda, y) + \langle \mathcal{J}(x_\lambda - x), y - x_\lambda\rangle \geq 0. \tag{1}$$

Note: for each $x \in X$, $x_\lambda \in D(A^F)$ and is uniquely defined. The operator $J_\lambda^F : X \to D(A^F)$ defined by $J_\lambda^F(x) = x_\lambda$ is called the resolvent of F. The resolvent solves the following: $\forall x \in X$, $\forall y \in C$,

$$\lambda F(J_\lambda^F(x), y) + \left\langle \mathcal{J}\left(J_\lambda^F(x) - x\right), y - J_\lambda^F(x)\right\rangle \geq 0.$$

Proposition 4 *Let $C \subseteq X$ be nonempty, closed and convex. If $F(\cdot, y)$ is upper hemicontinuous (i.e., upper semicontinuous on line segments) for all $y \in C$ and $F(x, \cdot)$ is convex and l.s.c. for all $x \in C$, then F is maximal monotone.*

Generalization:

Proposition 5 *Let $C \subseteq X$ be nonempty, closed and convex. Assume that $F(\cdot, y)$ is upper hemicontinuous for all $y \in C$ and $F(x, \cdot)$ is convex and l.s.c. for all $x \in C$. Further, let $G : C \times C \to \mathbb{R}$ be another monotone bifunction such that $G(\cdot, y)$ is u.s.c. for all $y \in C$, $G(x, \cdot)$ is convex for all $x \in C$, and for all $y \in C$, $\limsup_{\|x\| \to +\infty} \frac{G(x,y)}{\|x\|} < +\infty$. Then $F + G$ is maximal monotone.*

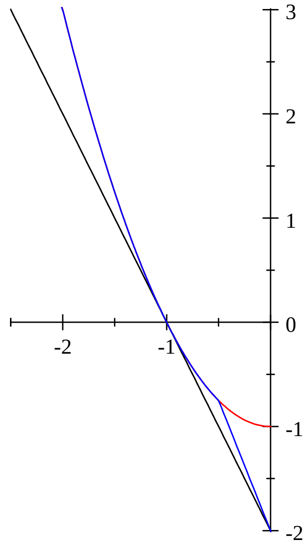
Example of a maximal monotone bifunction that is not included in the above proposition: $F : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}$ be given by $F(x,y) = x(x^2 - y^2)$. Then F is monotone and

$$A^F(x) = \begin{cases} \{-2x^2\}, \text{ if } x < 0 \\ \mathbb{R}^+, \text{ if } x = 0. \end{cases}$$

It is obvious that A^F is maximal monotone, thus F is maximal monotone. Define the bifunction $F_1 : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R}$ by

$$F_1(x,y) = \begin{cases} x(x^2 - y^2), & y < \frac{x}{2} \\ -\frac{5x^2}{2}y + 2x^3, & \frac{x}{2} \le y \le 0. \end{cases}$$

Then F_1 is monotone, $A^{F_1} = A^F$ thus F_1 is maximal monotone, but $F_1(x,\cdot)$ is not convex:



Graph of $F_1(-1, \cdot)$. The straight line is $-2x^2(\cdot - x)$ for $x = -1$.

3 Properties of the domain C.

For maximal monotone operators $T : X \to 2^{X^*}$, the domain $D(T)$ is known to have some properties. For instance, $\overline{D(T)}$ is convex. Some of these properties can be transferred to maximal monotone bifunctions.

Proposition 6 *Let $F : C \times C \to \mathbb{R}$ be maximal monotone. Assume that for every $x \in C$ and any converging sequence $\{x_n\} \subseteq C$, the sequence $\{F(x, x_n)\}$ is bounded from below. Then $C \subseteq \overline{D(A^F)}$. In particular, $\overline{C}$ is convex.*

Corollary 1 *Assume that C is closed, F is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is l.s.c.. Then C is convex and $C = \overline{D(A^F)}$.*

Corollary 2 *Assume that C is convex, F is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is convex and l.s.c.. Then $C \subseteq \overline{D(A^F)}$.*

If ϕ is a proper, convex, l.s.c. function, then $\operatorname{int}\operatorname{dom}\phi = \operatorname{int}\operatorname{dom}\partial\phi$. Here we have:

Proposition 7 *Assume that C is convex, and that $F(x,\cdot)$ is convex and l.s.c. for every $x \in C$. Then* $\operatorname{int} C = \operatorname{int} D(A^F)$.

An operator T is called locally bounded at $x_0 \in D(T)$ if there exists a neighborhood V of x_0 such that the set $T(V) = \bigcup_{x\in V} T(x)$ is bounded. We can generalize this definition to bifunctions:

Definition 2 *A bifunction F is called locally bounded at a point $x_0 \in X$ if there exists a neighborhood V of x_0 and $k \in \mathbb{R}$ such that $F(x,y) \leq k$ for all $x, y \in V \cap C$.*

Assuming local boundedness, we have:

Proposition 8 *Assume that C is convex and F is maximal monotone and locally bounded at every point of $\overline{C}$. Then $C \subseteq \overline{D(A^F)}$.*

Proposition 9 *Assume that F is maximal monotone and locally bounded on* $\operatorname{int} C$ *and that $C \subseteq \overline{D(A^F)}$. Then* $\operatorname{int} C = \operatorname{int} D(A^F)$.

What if we restrict the domain of F? Assume that $F : C \times C \to \mathbb{R}$ is maximal monotone and consider $K \subseteq C$. Let F_K be the restriction of F to K. The following proposition shows that maximality of F_K follows from a qualification condition:

Proposition 10 *Let $F : C \times C \to \mathbb{R}$ be maximal monotone and $K \subseteq C$ be a closed convex set such that $0 \in \operatorname{int}(K - D(A^F))$. Then F_K is maximal monotone and $A^{F_K}(x) = A^F(x) + N_K(x)$ for all $x \in K$, where*

$$N_K(x) = \{x^* \in X^* : \forall y \in K,\ \langle x^*, y - x\rangle \leq 0\}$$

is the normal cone to K at x.

4 Cyclically monotone bifunctions

Definition 3 *A bifunction F is called cyclically monotone if*

$$\forall x_1, x_2, \ldots x_n \in C, \quad F(x_1, x_2) + F(x_2, x_3) + \ldots + F(x_n, x_{n+1}) \leq 0$$

where $x_{n+1} := x_1$.

A necessary and sufficient condition for cyclic monotonicity:

Proposition 11 *A bifunction $F : C \times C \to \mathbb{R}$ is cyclically monotone if, and only if, there exists a function $f : C \to \mathbb{R}$ such that*

$$\forall x, y \in C, \quad F(x, y) \leq f(y) - f(x). \tag{2}$$

What if F is also maximal monotone?

Proposition 12 *Suppose that* int $C \neq \emptyset$ *and* $F : C \times C \to \mathbb{R}$ *is maximal monotone and cyclically monotone. Then the following statements are true:*

1) The sets $\overline{C}$ *and* int C *are convex, and equalities* $\overline{C} = \overline{D(A^F)}$ *and* int $C =$ int $D(A^F)$ *hold; the function* f *in relation (2) is uniquely defined up to a constant on* int C*, and is convex on* int C*.*

2) If in addition $F(x, \cdot)$ *is l.s.c. for every* $x \in C$*, then* f *is uniquely defined up to a constant, and convex and l.s.c. on* C*.*

If $F(x, \cdot)$ is not l.s.c., a convex function such that (2) holds may not exist.

Example. Let $\phi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be the function with $\mathbf{dom}(\phi) = Q := [-1, 1] \times [-1, 1]$ defined by

$$\phi(a, b) = -\left(\sqrt{1 - a^2} + \sqrt{1 - b^2}\right).$$

Let further f be any function with $\mathbf{dom}(f) = Q$ and such that $f = \phi$ on $\mathbf{int}\, Q$, $f \geq \phi$ on ∂Q, and f is not convex (for instance, $f = \phi$ on $\mathbb{R}^2 \backslash \{(1, 0)\}$ and $f(1, 0) = 1$). Define the cyclically monotone bifunction $F : Q \times Q \to \mathbb{R}$ by

$$F(x, y) = f(y) - f(x).$$

Then F is maximal monotone and cyclically monotone, but there does not exist any convex function $f_1 : Q \to \mathbb{R}$ such that

$$\forall x, y \in Q,\ F(x, y) \leq f_1(y) - f_1(x).$$

If F is cyclically monotone, then A^F is also cyclically monotone. The next example shows that cyclic monotonicity of A^F does not imply cyclic monotonicity of F, even if F is monotone, C is a convex subset of $\mathbb{R}$ and A^F is a subdifferential of a proper l.s.c. convex function.

Example. Let $C = (-1, 1]$ and $f(x) = \frac{1}{1-x^2}$ for $x \in (-1, 1)$. Define $F : C \times C \to \mathbb{R}$ by

$$F(x, y) = f(y) - f(x)$$

whenever $x, y \in (-1, 1)$, $F(x, 1) = f'(x)(1 - x)$ and $F(1, x) = -f'(x)(1-x)$ for $x \in (-1, 1)$, and $F(1, 1) = 0$. It is obvious that F is monotone. It is easy to see that $A^F = \partial f$ and in particular A^F is cyclically monotone. However, F is not cyclically monotone since

$$F(\frac{-1}{2}, \frac{1}{2}) + F(\frac{1}{2}, 1) + F(1, \frac{-1}{2}) =$$
$$0 + f'(\frac{1}{2})(1 - \frac{1}{2}) - f'(\frac{-1}{2})(1 + \frac{1}{2}) = 2f'(\frac{1}{2}) > 0.$$