Maximal Monotonicity of Bifunctions

Nicolas Hadjisavvas
University of the Aegean, Greece

Hadi Khatibzadeh
Tarbiat Modares University, Iran
1 Monotone Bifunctions

$X$: Banach space; $C \subseteq X$ nonempty.

Bifunction: any function $F : C \times C \rightarrow \mathbb{R}$ such that $F(x, x) = 0$, $\forall x \in C$.

Equilibrium problem: Find $x_0 \in C$ such that

$$\forall y \in C, \quad F(x_0, y) \geq 0.$$  \hspace{1cm} \text{(EP)}

Example: Given a multivalued operator $T : C \rightarrow 2^{X^* \setminus \{\emptyset\}}$ with weak*–compact values, set

$$F(x, y) = \max_{x^* \in T(x)} \langle x^*, y - x \rangle.$$ 

Then $F$ is a bifunction ($F(x, x) = 0$) and (EP) is equivalent to the variational inequality problem: find $x_0 \in C$ such that

$$\forall y \in C, \exists x^* \in T(x_0), \quad \langle x^*, y - x \rangle \geq 0. \quad \text{(VIP)}$$

Other examples: Mathematical Programming problem, fixed point problem, saddle point problem...
\( T : X \to 2^{X^*} \) an operator.

\( D(T) = \{ x \in X : T(x) \neq \emptyset \} \) the domain of \( T \).

\( \text{gr } T = \{(x, x^*) \in X \times X^* : x^* \in T(x)\} \) the graph of \( T \).

\( T \) is called monotone if

\[ \forall (x, x^*), (y, y^*) \in \text{gr } T : \langle y^* - x^*, y - x \rangle \geq 0. \]

A monotone operator \( T \) is called maximal monotone if there is no monotone operator \( S \) with \( \text{gr } T \subsetneq \text{gr } S \).

A bifunction \( F \) is called monotone if

\[ \forall x, y \in C, \quad F(x, y) + F(y, x) \leq 0. \]

Example: If \( T : X \to 2^{X^*} \) is any monotone operator, define the bifunction \( G_T : D(T) \times D(T) \to \mathbb{R} \) by

\[ G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle. \]

Since \( T \) is monotone, \( G_T \) is also monotone.
Starting from a monotone bifunction $F : C \times C \to \mathbb{R}$, one can define a monotone operator $A^F$ as follows: for any $x \in C$ set

$$A^F(x) = \{x^* \in X^* : \forall y \in C, F(x, y) \geq \langle x^*, y - x \rangle\}$$

while $A^F(x) = \emptyset$ if $X \setminus C$.

If $\hat{F} : C \times X \to \mathbb{R} \cup \{+\infty\}$ is the extension of $F$ defined by setting $\hat{F}(x, y) = +\infty$ for $x \in C$, $y \in X \setminus C$, then

$$\forall x \in C, \quad A^F(x) = \partial \hat{F}(x, \cdot)(x).$$

Special case: if $f : X \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom } f = C$ and $F(x, y) = f(y) - f(x)$, then $F$ is monotone and $A^F = \partial f$. In more general cases, $A^F$ is not the subdifferential of a unique function. However, it is monotone (same proof as for the subdifferential), and $A^F(x)$ is closed and convex.

Note: Different bifunctions $F_1$ and $F_2$ on $C$ may produce the same operator: $A^{F_1} = A^{F_2}$. 
Definition 1 A monotone bifunction $F$ is called maximal monotone, if $A^F$ is maximal monotone.

Another definition of maximal monotonicity (Blum-Oettli): $F$ is BO-maximal monotone if for every $x \in C$, $x^* \in X^*$ the following implication holds:

$$\forall y \in C, F(y, x) + \langle x^*, y - x \rangle \leq 0$$

$$\Rightarrow \forall y \in C, F(x, y) \geq \langle x^*, y - x \rangle.$$ 

If $C$ is convex and $F(x, \cdot)$ is convex and l.s.c., the two definitions of maximal monotonicity are equivalent (Ait Mansour, Chbani and Riahi, 2003). This is not the case in general.

Example: $f : X \to \mathbb{R}$ any odd function, $F(x, y) = f(y - x)$. Then $F$ is monotone, and in fact BO-maximal monotone. One can check: $F$ is maximal monotone if and only if $f \in X^*$. In fact, if $f \notin X^*$, then $D(A^F) = \emptyset$. For instance, $F(x, y) = (y - x)^3$ is BO-maximal monotone, but not maximal monotone.
**Proposition 1** If the operator $T$ is maximal monotone, then $G_T$ is maximal monotone. In fact, $A^{G_T} = T$.

The converse does not hold: If $G_T$ is maximal monotone, then $T$ is not necessarily maximal monotone, as it is possible to have $T \neq S$ and $G_T = G_S$. For instance, if $T$ is maximal monotone and $S \neq T$ but $\overline{\text{co}}S(x) = T(x)$ for all $x \in X$, then $G_S = G_T$ hence $G_S$ is maximal monotone, while $S$ is not. Another example: $T$ is defined on $\mathbb{R}$, $D(T) = [0, +\infty)$, $T(x) = \{0\}$ for $x \in [0, +\infty)$. Then $G_T$ is maximal, $T$ is not maximal.

**Proposition 2** Assume that $T$ has closed convex values and $D(T) = X$. If $G_T$ is maximal monotone, then $T$ is maximal monotone.

Starting from a maximal monotone bifunction $F$, one may construct $A^F$ and then the maximal monotone bifunction $G := G_{A^F}$. In general, $G(x, y) \leq F(x, y)$ for $x, y \in D(A^F)$, $A^F = A^G$, but $F \neq G$. Example: $F(x, y) = y^2 - x^2$, then $A^F(x) = \{2x\}$, $G(x) = 2x(y - x)$. 
2 Conditions for maximality of bifunctions

In the following, $X$ will be reflexive, equipped with a norm such that both $X$ and $X^*$ are strictly convex ($\|x\| + \|y\| = \|x + y\|$ implies that $x$, $y$ are proportional). Then for each $x \in X$ there exists a unique $x^* \in X^*$ such that $\langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2$. The correspondence $x \to J(x) := x^*$ is the so-called duality map.

The following proposition is a consequence of the results of Ait Mansour/Chbani/Riahi, 2003.

**Proposition 3** A monotone bifunction $F$ is maximal monotone if and only if for each $\lambda > 0$ (equivalently, for some $\lambda > 0$) and each $x \in X$ there exists $x_\lambda \in C$ such that

$$\forall y \in C, \quad \lambda F(x_\lambda, y) + \langle J(x_\lambda - x), y - x_\lambda \rangle \geq 0.$$  

(1)
Note: for each $x \in X$, $x_\lambda \in D(A^F)$ and is uniquely defined. The operator $J^F_\lambda : X \to D(A^F)$ defined by $J^F_\lambda(x) = x_\lambda$ is called the resolvent of $F$. The resolvent solves the following: $\forall x \in X$, $\forall y \in C$,

$$\lambda F(J^F_\lambda(x), y) + \langle J(J^F_\lambda(x) - x), y - J^F_\lambda(x) \rangle \geq 0.$$ 

**Proposition 4** Let $C \subseteq X$ be nonempty, closed and convex. If $F(\cdot, y)$ is upper hemicontinuous (i.e., upper semicontinuous on line segments) for all $y \in C$ and $F(x, \cdot)$ is convex and l.s.c. for all $x \in C$, then $F$ is maximal monotone.

Generalization:

**Proposition 5** Let $C \subseteq X$ be nonempty, closed and convex. Assume that $F(\cdot, y)$ is upper hemicontinuous for all $y \in C$ and $F(x, \cdot)$ is convex and l.s.c. for all $x \in C$. Further, let $G : C \times C \to \mathbb{R}$ be another monotone bifunction such that $G(\cdot, y)$ is u.s.c. for all $y \in C$, $G(x, \cdot)$ is convex for all $x \in C$, and for all $y \in C$, $\limsup_{\|x\| \to +\infty} \frac{G(x, y)}{\|x\|} < +\infty$. Then $F + G$ is maximal monotone.
Example of a maximal monotone bifunction that is not included in the above proposition: \( F : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R} \) be given by \( F(x, y) = x(x^2 - y^2) \). Then \( F \) is monotone and

\[
A^F(x) = \begin{cases} 
-2x^2, & \text{if } x < 0 \\
\mathbb{R}^+, & \text{if } x = 0.
\end{cases}
\]

It is obvious that \( A^F \) is maximal monotone, thus \( F \) is maximal monotone. Define the bifunction \( F_1 : \mathbb{R}^- \times \mathbb{R}^- \to \mathbb{R} \) by

\[
F_1(x, y) = \begin{cases} 
x(x^2 - y^2), & y < \frac{x}{2} \\
-\frac{5x^2}{2}y + 2x^3, & \frac{x}{2} \leq y \leq 0.
\end{cases}
\]

Then \( F_1 \) is monotone, \( A^{F_1} = A^F \) thus \( F_1 \) is maximal monotone, but \( F_1(x, \cdot) \) is not convex:
Graph of $F_1(-1, \cdot)$. The straight line is $-2x^2(\cdot - x)$ for $x = -1$. 
3 Properties of the domain $C$.

For maximal monotone operators $T : X \to 2^{X^*}$, the domain $D(T)$ is known to have some properties. For instance, $\overline{D(T)}$ is convex. Some of these properties can be transferred to maximal monotone bifunctions.

**Proposition 6** Let $F : C \times C \to \mathbb{R}$ be maximal monotone. Assume that for every $x \in C$ and any converging sequence $\{x_n\} \subseteq C$, the sequence $\{F(x, x_n)\}$ is bounded from below. Then $C \subseteq \overline{D(A^F)}$. In particular, $\overline{C}$ is convex.

**Corollary 1** Assume that $C$ is closed, $F$ is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is l.s.c.. Then $C$ is convex and $C = \overline{D(A^F)}$.

**Corollary 2** Assume that $C$ is convex, $F$ is maximal monotone, and that for every $x \in C$, $F(x, \cdot)$ is convex and l.s.c.. Then $C \subseteq \overline{D(A^F)}$. 
If $\phi$ is a proper, convex, l.s.c. function, then $\text{int dom } \phi = \text{int dom } \partial \phi$. Here we have:

**Proposition 7** Assume that $C$ is convex, and that $F(x, \cdot)$ is convex and l.s.c. for every $x \in C$. Then $\text{int } C = \text{int } D(A^F)$.

An operator $T$ is called locally bounded at $x_0 \in D(T)$ if there exists a neighborhood $V$ of $x_0$ such that the set $T(V) = \bigcup_{x \in V} T(x)$ is bounded. We can generalize this definition to bifunctions:

**Definition 2** A bifunction $F$ is called locally bounded at a point $x_0 \in X$ if there exists a neighborhood $V$ of $x_0$ and $k \in \mathbb{R}$ such that $F(x, y) \leq k$ for all $x, y \in V \cap C$.

Assuming local boundedness, we have:
**Proposition 8** Assume that $C$ is convex and $F$ is maximal monotone and locally bounded at every point of $\overline{C}$. Then $C \subseteq D(A^F)$.

**Proposition 9** Assume that $F$ is maximal monotone and locally bounded on $\text{int } C$ and that $C \subseteq D(A^F)$. Then $\text{int } C = \text{int } D(A^F)$.

What if we restrict the domain of $F$? Assume that $F : C \times C \to \mathbb{R}$ is maximal monotone and consider $K \subseteq C$. Let $F_K$ be the restriction of $F$ to $K$. The following proposition shows that maximality of $F_K$ follows from a qualification condition:

**Proposition 10** Let $F : C \times C \to \mathbb{R}$ be maximal monotone and $K \subseteq C$ be a closed convex set such that $0 \in \text{int}(K - D(A^F))$. Then $F_K$ is maximal monotone and $A^{F_K}(x) = A^F(x) + N_K(x)$ for all $x \in K$, where

$$N_K(x) = \{x^* \in X^* : \forall y \in K, \langle x^*, y - x \rangle \leq 0\}$$

is the normal cone to $K$ at $x$. 
4 Cyclically monotone bifunctions

Definition 3 A bifunction $F$ is called cyclically monotone if

$$\forall x_1, x_2, \ldots x_n \in C, \quad F(x_1, x_2) + F(x_2, x_3) + \ldots + F(x_n, x_{n+1}) \leq 0$$

where $x_{n+1} := x_1$.

A necessary and sufficient condition for cyclic monotonicity:

Proposition 11 A bifunction $F : C \times C \rightarrow \mathbb{R}$ is cyclically monotone if, and only if, there exists a function $f : C \rightarrow \mathbb{R}$ such that

$$\forall x, y \in C, \quad F(x, y) \leq f(y) - f(x). \quad (2)$$

What if $F$ is also maximal monotone?
Proposition 12 Suppose that \( \text{int } C \neq \emptyset \) and \( F : C \times C \rightarrow \mathbb{R} \) is maximal monotone and cyclically monotone. Then the following statements are true:

1) The sets \( \overline{C} \) and \( \text{int } C \) are convex, and equalities \( \overline{C} = D(A^F) \) and \( \text{int } C = \text{int } D(A^F) \) hold; the function \( f \) in relation (2) is uniquely defined up to a constant on \( \text{int } C \), and is convex on \( \text{int } C \).

2) If in addition \( F(x, \cdot) \) is l.s.c. for every \( x \in C \), then \( f \) is uniquely defined up to a constant, and convex and l.s.c. on \( C \).

If \( F(x, \cdot) \) is not l.s.c., a convex function such that (2) holds may not exist.
Example. Let $\phi : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ be the function with $\text{dom}(\phi) = Q := [-1, 1] \times [-1, 1]$ defined by

$$
\phi(a, b) = - \left( \sqrt{1 - a^2} + \sqrt{1 - b^2} \right).
$$

Let further $f$ be any function with $\text{dom}(f) = Q$ and such that $f = \phi$ on $\text{int} Q$, $f \geq \phi$ on $\partial Q$, and $f$ is not convex (for instance, $f = \phi$ on $\mathbb{R}^2 \setminus \{(1, 0)\}$ and $f(1, 0) = 1$). Define the cyclically monotone bifunction $F : Q \times Q \to \mathbb{R}$ by

$$
F(x, y) = f(y) - f(x).
$$

Then $F$ is maximal monotone and cyclically monotone, but there does not exist any convex function $f_1 : Q \to \mathbb{R}$ such that

$$
\forall x, y \in Q, \ F(x, y) \leq f_1(y) - f_1(x).
$$
If $F$ is cyclically monotone, then $A^F$ is also cyclically monotone. The next example shows that cyclic monotonicity of $A^F$ does not imply cyclic monotonicity of $F$, even if $F$ is monotone, $C$ is a convex subset of $\mathbb{R}$ and $A^F$ is a subdifferential of a proper l.s.c. convex function.

**Example.** Let $C = (-1, 1]$ and $f(x) = \frac{1}{1-x^2}$ for $x \in (-1, 1)$. Define $F : C \times C \to \mathbb{R}$ by

$$F(x, y) = f(y) - f(x)$$

whenever $x, y \in (-1, 1)$, $F(x, 1) = f'(x)(1-x)$ and $F(1, x) = -f'(x)(1-x)$ for $x \in (-1, 1)$, and $F(1, 1) = 0$. It is obvious that $F$ is monotone. It is easy to see that $A^F = \partial f$ and in particular $A^F$ is cyclically monotone. However, $F$ is not cyclically monotone since

$$F\left(\frac{-1}{2}, \frac{1}{2}\right) + F\left(\frac{1}{2}, 1\right) + F(1, \frac{-1}{2}) =$$

$$0 + f'\left(\frac{1}{2}\right)(1 - \frac{1}{2}) - f'\left(\frac{-1}{2}\right)(1 + \frac{1}{2}) = 2f'\left(\frac{1}{2}\right) > 0.$$