Some properties of the Variational Sum of Monotone Operators

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Outline

Basic notions

Let $X \neq \{0\}$ be a reflexive real Banach space, X^* its dual.
$\langle \cdot, \cdot \rangle : X^* \times X \longrightarrow \mathbb{R}$ the duality pairing.
A multivalued operator $T : X \rightrightarrows X^*$ is called

- *Monotone*, if

$$\langle y^* - x^*, y - x \rangle \geq 0, \ \forall (x, x^*), (y, y^*) \in \mathrm{Gr}(T).$$

where

$$\mathrm{Gr}(T) := \left\{ (x, x^*) \in X \times X^* : x^* \in Tx \right\}.$$

- *Maximal monotone*, if
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- *Maximal monotone*, if
  $\mathrm{Gr}(T)$ is not contained properly in the graph of any other monotone operator between X and X^*.

- *Pre-maximal monotone*, if
  it has a unique maximal monotone extension.

Examples of monotone operators

- Let $T : X \longrightarrow X^*$ be a linear operator, then

$$T \text{ monotone} \Leftrightarrow T \text{ is positive } \left(\langle Tx, x\rangle \geq 0,\ \forall x \in X\right).$$

- $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, $\operatorname{dom} f := \{x \in X : f(x) < +\infty\} \neq \emptyset$.
  The subdifferential of f, $\partial f : X \rightrightarrows X^*$, is
  $\partial f(x) := \left\{x^* \in X^* : f(y) \geq f(x) + \langle x^*, y - x\rangle,\ \forall y \in X\right\}.$

$$f \text{ proper, lsc, convex.} \Rightarrow \partial f \text{ maximal monotone}$$

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- Let $T : \mathbb{R} \rightrightarrows \mathbb{R}$ the operator defined by $Tx = \{x\}$ if $x < 0$ and $Tx = \{x+1\}$ if $x > 0$.

T is a pre-maximal monotone operator.

◇ Motivation

Theorem (Brezis, 1973)

Let $H \neq \{0\}$ be a real Hilbert space, $T_1 : H \rightrightarrows H$ a maximal monotone operator and $T_2 : H \to H$ a Lipschitz monotone operator. Then,

$$T_1 + T_2 \text{ is a maximal monotone operator.}$$

Facts:

- $T : H \rightrightarrows H$ maximal monotone and $\lambda > 0$.
  The *Yosida regularization* $T_\lambda : H \to H$ is a ($\mathrm{DomT}_\lambda = \mathrm{H}$, single-valued) Lipschitz monotone operator .
- If $T_1, T_2 : H \rightrightarrows H$ are maximal monotones, for $\lambda, \mu \geq 0, \lambda + \mu > 0$,
  $$T_{1,\lambda} + T_{2,\mu} \text{ is a maximal monotone operator.}$$

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Tools

The Yosida Regularization

Let $J : X \rightrightarrows X^*$ be the duality operator (maximal monotone), $J(x) = \{x^* : \|x\|^2 = \|x^*\|^2 = \langle x^*, x\rangle\}$ and $\mathrm{Dom}(J) = X$.
$T : X \rightrightarrows X^*$ maximal monotone operator, and let $\lambda > 0$.
Given $x \in X \quad \exists! \quad x_\lambda = J_\lambda^T(x)$ solution of the inclusion

$$0 \in J(x_\lambda - x) + \lambda T(x_\lambda).$$

The *Yosida regularization* of T of order $\lambda > 0$ is the operator $T_\lambda : X \to X^*$ defined by

$$T_\lambda(x) = \frac{1}{\lambda} J(x - x_\lambda),\ x \in X.$$

By convention we put $T_0 = T$.

$T : X \rightrightarrows X$ maximal monotone and $\lambda > 0$.

Facts:

- For $\lambda > 0$, T_λ is single-valued max. mon. with $\mathrm{Dom} T = X$.
- T_λ is norm-to-weak continuous.
- $\|T_\lambda(x)\| \leq \frac{1}{\lambda}\|x\|, \forall x \in X$.
- $\|T_\lambda(x)\| \leq \|T^{\min}(x)\|, \forall x \in \mathrm{Dom} T$.

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Definition (Attouch-Baillon-Théra†, 1993 / Revalski-Théra‡, 1999)

Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. The *variational sum* of T_1 and T_2,$T_1 \underset{v}{+} T_2 : X \rightrightarrows X^*$, is the operator defined by

$$T_1 \underset{v}{+} T_2 = PK - \liminf_{(\lambda,\mu)\in I} (T_{1,\lambda} + T_{2,\mu}).$$

Equivalently, $(x, x^*) \in (T_1 \underset{v}{+} T_2) \Leftrightarrow$ for each

$$\{(\lambda_n, \mu_n)\} \in I = \{\{(\lambda_n, \mu_n)\} : \lambda_n, \mu_n \geq 0, \lambda_n + \mu_n > 0, \lambda_n, \mu_n \to 0\},$$

$\exists\{(x_n, x_n^*)\} \subset X \times X^*$, such that

$$\forall n \in \mathbb{N}, (x_n, x_n^*) \in (T_{1,\lambda_n} + T_{2,\mu_n}), \text{ and}$$

$$(x_n, x_n^*) \to (x, x^*).$$

† In the setting of real Hilbert spaces.

‡ In the setting of arbitrary reflexive real Banach spaces.

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Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then

(i) $T_1 \underset{v}{+} T_2$ is monotone.

(ii) $\mathrm{Dom}(T_1 + T_2) \subset \mathrm{Dom}(T_1 \underset{v}{+} T_2)$.

(iii) $T_1 \underset{v}{+} T_2$ maximal monotone $\Rightarrow \mathrm{Gr}(T_1 + T_2) \subset \mathrm{Gr}(T_1 \underset{v}{+} T_2)$.

(iv) $T_1 + T_2$ maximal monotone $\Rightarrow T_1 + T_2 = T_1 \underset{v}{+} T_2$.

(v) f, g proper, lsc, convex, $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset \Rightarrow$

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Schemes of proofs:

(iv) Let $y^* \in X^*$. For $\{(\lambda_n, \mu_n)\} \in I$, we consider y_{λ_n,μ_n} solution of

$$y^* \in J(y_{\lambda_n,\mu_n}) + T_{1,\lambda_n}(y_{\lambda_n,\mu_n}) + T_{2,\mu_n}(y_{\lambda_n,\mu_n}).$$

Objective: To show that $y_{\lambda_n,\mu_n} \to y$, where y is the only solution of

$$y^* \in J(y) + (T_1 + T_2)(y).$$

(iii) Let $(y, y^*) \in X^*$. For $\{(\lambda_n, \mu_n)\} \in I$, we consider y_{λ_n,μ_n} solution of

$$y^* + Jy \in J(y_{\lambda_n,\mu_n}) + T_{1,\lambda_n}(y_{\lambda_n,\mu_n}) + T_{2,\mu_n}(y_{\lambda_n,\mu_n}).$$

i.e. $y^* + Jy = J(y_{\lambda_n,\mu_n}) + y^*_{\lambda_n,\mu_n}$, with $y^*_{\lambda_n,\mu_n} \in \left(T_{1,\lambda_n} + T_{2,\mu_n}\right)(y_{\lambda_n,\mu_n})$.

Objective: If $(y, y^*) \in \mathrm{Gr}(T_1 + T_2)$ to show that

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A new approach

Given two maximal monotone operators $T_1, T_2 : X \rightrightarrows X^*$ and $_b igl\{(\lambda_n, \mu_n)\} \in I$, for each $(x, x^*) \in X \times X^*$ and $n \in \mathbb{N}$, we denote by $\psi_{\lambda_n,\mu_n}(x, x^*) = x_n$ the unique solution of

$$x^* \in J(x_n - x) + T_{1,\lambda_n}(x_n) + T_{2,\mu_n}(x_n). \quad (1)$$

Proposition

Let $T_1, T_2 : X \rightrightarrows X^*$ *be two maximal monotone operators and let* $(y, y^*) \in X \times X^*$. *The following are equivalent:*

(i) $(y, y^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$;

(ii) *For every* $(x, x^*) \in X \times X^*$ *and* $\{(\lambda_n, \mu_n)\} \in I$, *the sequence* $\{x_n\}$ *where* $x_n = \psi_{\lambda_n, \mu_n}(x, x^*)$ *for each* $n \in \mathbb{N}$, *is bounded and for every subsequence* $x_{n_k} \rightharpoonup \overline{x}$

$$\frac{1}{2}\|y - x\|^2 + \langle x^* - y^*, \overline{x} - y \rangle \geq \frac{1}{2} \limsup \|x_{n_k} - x\|^2. \tag{2}$$

(iii) *For any* $\{(\lambda_n, \mu_n)\} \subset I$, $\psi_{\lambda_n, \mu_n}(y, y^*) \to y$.

Proposition

Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then:

(i) *If* $\{(y_n, y_n^*)\} \subset \mathrm{Gr}(T_1 \underset{v}{+} T_2)$, $y_n \longrightarrow y$ *and* $y_n \rightharpoonup y^*$, *then* $(y, y^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2)$.

(ii) $T_1 \underset{v}{+} T_2$ *has closed graph and convex values.*

(iii) $(T_1 \underset{v}{+} T_2)^{-1}$ *has closed-convex values.*

Theorem

Let X be an Euclidean space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then $T_1 \underset{v}{+} T_2$ coincides with the intersection of all its maximal monotone extensions.

Corollary

Let X be an Euclidean space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $T_1 \underset{v}{+} T_2$ is premaximal, then it is maximal.

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$$\exists \{(x_n, x_n^*)\} \subset X \times X^* : \forall\, n \in \mathbb{N}, (x_n, x_n^*) \in T_{1,\lambda_n} + T_{2,\mu_n},$$
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Equivalently:

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Properties:

- $w \times w - T_1 \underset{v}{+} T_2$ is a monotone operator.
- If X is an Euclidian space, $T_1 \underset{v}{+} T_2 = w \times w - T_1 \underset{v}{+} T_2$.

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Theorem

Let X be a Hilbert space, and let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then

1. $w \times w - T_1 \underset{v}{+} T_2$ *coincides with the intersection of all its maximal monotone extensions.*
2. *If* $(w \times w-)T_1 \underset{v}{+} T_2$ *is pre-maximal, then* $w \times w - T_1 \underset{v}{+} T_2$ *is the unique maximal monotone extension.*

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Relationship with the usual sum

Theorem

Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. Then

$$\mathrm{Gr}(T_1 + T_2) \subset \mathrm{Gr}(T_1 \underset{v}{+} T_2).$$

Corollary

Let $T_1, T_2 : X \rightrightarrows X^*$ be two maximal monotone operators. If $T_1 + T_2$ is a maximal monotone operator, then

$$T_1 + T_2 = T_1 \underset{v}{+} T_2.$$

Corollary

Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be two proper lower semicontinuous convex functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$. Then

$$\partial(f + g) = \partial f \underset{v}{+} \partial g.$$

Extended Sum

Definition (Revalski-Théra, 1999)

Let $T_1, T_2 : X \rightrightarrows X^*$ be two monotone operators. The *extended sum* of T_1 and T_2 at the point $x \in X$ is defined by

$$(T_1 \underset{\text{ext}}{+} T_2)(x) = \bigcap_{\epsilon > 0} \overline{T_1^\epsilon x + T_2^\epsilon x}^{w^*}$$

where, given $\epsilon \geq 0$, the $\epsilon-$*enlargement* of T, $T^\epsilon : X \rightrightarrows X^*$ is defined by:

$$T^\epsilon x = \{x^* \in X^* : \langle y^* - x^*, y - x \rangle \geq -\varepsilon, \forall (y, y^*) \in \operatorname{Gr}(T)\}.$$

Facts:

- $\mathrm{Gr}(T_1 + T_2) \subset \mathrm{Gr}(T_1 \underset{\text{ext}}{+} T_2)$.
- $T_1 + T_2$ maximal monotone $\Rightarrow T_1 + T_2 = T_1 \underset{\text{ext}}{+} T_2$.
- f, g proper lsc convex, $\mathrm{dom}\, f \cap \mathrm{dom}\, g \neq \emptyset \Rightarrow$

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Facts:

- $\mathrm{Gr}(T_1 + T_2) \subset \mathrm{Gr}(T_1 \underset{\text{ext}}{+} T_2)$.
- $T_1 + T_2$ maximal monotone $\Rightarrow T_1 + T_2 = T_1 \underset{\text{ext}}{+} T_2$.
- f, g proper lsc convex, $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset \Rightarrow$

$$\partial(f+g)(x) = (\partial f \underset{\text{ext}}{+} \partial g)(x),\ \forall\, x \in X.$$

Relationship between the Extended Sum and the Variational Sum

Proposition (Revalski-Théra)

Let $T_1, T_2 : X \rightrightarrows X^*$ *two maximal monotone operators. Then,*

$$\langle u^* - v^*, u - v\rangle \geq 0, \forall (u, u^*) \in \mathrm{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2), (v, v^*) \in \mathrm{Gr}(T_1 \underset{v}{+} T_2).$$

As a consequence:

- *If* $T_1 \underset{\mathrm{ext}}{+} T_2$ *max. mon.*, $\Rightarrow \mathrm{Gr}(T_1 \underset{v}{+} T_2) \subset \mathrm{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2)$.
- *If* $T_1 \underset{v}{+} T_2$ *max. mon.*, $\Rightarrow \mathrm{Gr}(T_1 \underset{\mathrm{ext}}{+} T_2) \subset \mathrm{Gr}(T_1 \underset{v}{+} T_2)$.

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Extended vs Variational

Example

$X = l_2$, $D = \{\{x_n\} \subset l_2 : \{2^n x_n\} \in l_2\}$, $D \subsetneq l_2$, $\overline{D} = l_2$.

Let $T : D \times D \to l_2 \times l_2$, $T(\{x_n\}, \{y_n\}) = (\{2^n y_n\}, -\{2^n x_n\})$.

$T_1 = T$ and $T_2 = -T$ are two maximal monotone operator.

$$T_1 \underset{\text{ext}}{+} T_2 \text{ is not maximal BUT } \overline{T_1 \underset{\text{ext}}{+} T_2}^{G} = T_1 \underset{v}{+} T_2 \text{ is it.}$$

Mean Result about the Relationship

Theorem

Let $T_1, T_2 : X \rightrightarrows X^*$ *be two maximal monotone operators. Then*

$$\operatorname{Gr}(T_1 \underset{\text{ext}}{+} T_2) \subset \operatorname{Gr}(T_1 \underset{v}{+} T_2).$$

Open questions

- $w \times w - T_1 \underset{v}{+} T_2 = T_1 \underset{v}{+} T_2$??
- X Banach reflexive: $w \times w - T_1 \underset{v}{+} T_2$??
- $T_1 \underset{v}{+} T_2$ is maximal ??

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Thanks for your attention!!