Viable harvest of monotone bioeconomics models

Preservation and production issues

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Outline

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The core model for fishery management decisions
- The model
- Some questions about sustainability of landings

Discrete time viability issues
- Monotonicity properties

Viability kernel properties
- Maximum sustainable yield
- Minimal viable feedback

The Patagonian toothfish (Légine australe)
- Questions & Answers

Age structured model

- the state : $N = (N_a)_{a=1,\dots,A} \in \mathbb{R}_+^A$, the *abundances* at age
- the control : λ the *fishing effort multiplier*
- the dynamics : $N(t+1) = g(N(t), \lambda(t))$ given by

$$\begin{cases} g_1(N,\lambda) &= \varphi(SSB(N))\,, \\ g_a(N,\lambda) &= e^{-(M_{a-1}+\lambda F_{a-1})} N_{a-1}\,, \quad a = 2,\dots,A-1\,, \\ g_A(N,\lambda) &= e^{-(M_{A-1}+\lambda F_{A-1})} N_{A-1} + e^{-(M_A+\lambda F_A)} N_A\,. \end{cases}$$

where

- the *spawning stock biomass SSB* is defined by

$$SSB(N) = \sum_{a=1}^{A} \gamma_a \upsilon_a N_a$$

- the function φ describes the *stock-recruitment relationship*
- M_a is the natural *mortality rate* of individuals of age a
- F_a is the mortality rate of individuals of age a due to harvesting

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Harvested fish population age structured model

Example

Typical stock-recruitment relationship:

- *Constant:* $\varphi(B) = R$.
- *Linear:* $\varphi(B) = RB$.
- *Beverton-Holt:* $\varphi(B) = \frac{B}{\alpha + \beta B}$.
- *Ricker:* $\varphi(B) = \alpha B e^{-\beta B}$.

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The harvest term

The exploitation is described by catch-at-age C_a and yield Y, both defined for a given vector of abundance N and a given control λ :

$$C_a(N,\lambda) = \frac{\lambda F_a}{\lambda F_a + M_a}\left(1 - e^{-(M_a+\lambda F_a)}\right) N_a$$

The production in term of biomass is :

$$Y(N,\lambda) = \sum_{a=1}^{A} w_a\, C_a(N,\lambda)$$

Questions

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1,..,A}$ (initial conditions) for which one can always harvest at least that quantity?
- What levels of catch (landings) are non sustainable?
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Discrete time control system

Let us consider a nonlinear control system described in discrete time by the difference equation

$$\begin{cases} N(t+1) = g(N(t), \lambda(t)), & \forall t \in \mathbb{N}, \\ N_0 \quad \text{given}, & \end{cases}$$

where

- The state variable $N(t)$ belongs to the state space $\mathbb{X} \subseteq R^{n_{\mathbb{X}}}$.
- The control variable $\lambda(t)$ is an element of the control set $\mathbb{U} \subseteq R^{n_{\mathbb{U}}}$.
- The dynamics g maps $\mathbb{X} \times \mathbb{U}$ into $\mathbb{X}$.

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Desirable configurations

A decision maker describes desirable configurations of the system through a set $\mathbb{D} \subset \mathbb{X} \times \mathbb{U}$ termed the desirable set

$$(N(t), \lambda(t)) \in \mathbb{D}, \quad \forall t \in \mathbb{N},$$

where $\mathbb{D}$ includes both system states and controls constraints.

Example

- $\mathbb{D}_{\text{protect}} := \{(N, \lambda) \ : \ N \geq \bar{N}\}$
- $\mathbb{D}_{\text{yield}} := \{(N, \lambda) \ : \ Y(N, \lambda) \geq y_{\min}, \quad SSB(N) \geq B_{\lim}\}$
- $\mathbb{D}_{ICES} := \{(N, \lambda) \ : \ SSB(N) \geq B_{\lim}, \ F(\lambda) \leq F_{\lim}\}$

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Viability domains and viability kernel

Definition

- $\mathbb{V} \subset \mathbb{X}$ *is a* *Viability Domain* *if for all* $N \in \mathbb{V}$ *there exists* $\lambda \in \mathbb{U}$ *such that* $(N, \lambda) \in \mathbb{D}$ *and* $g(N, \lambda) \in \mathbb{V}$.
- *Viability kernel*

$$\mathbb{V}(g, \mathbb{D}) = \left\{ \begin{array}{l} N_0 \in \mathbb{X} \ : \ \textit{there exist} \ \ \lambda(0), \lambda(1), \lambda(2), \ldots \\ N(0), N(1), N(2), \ldots \textit{such that } N(0) = N_0 \\ N(t+1) = g(N(t), \lambda(t)) \ \ \textit{and} \\ (N(t), \lambda(t)) \in \mathbb{D} \end{array} \right.$$

Goals

- *Determine or approximate the viability kernel* $\mathbb{V}(g, \mathbb{D})$ *for a given dynamics* g *and a given desirable set* $\mathbb{D}$.
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Monotonicity properties on the dynamics

Definition

We say that the dynamics $g : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ *is a monotone bioeconomic dynamics if g is increasing with respect to the state i.e.*

$$\forall (N, N', \lambda) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}, \quad N' \geq N \Rightarrow g(N', \lambda) \geq g(N, \lambda),$$

and is decreasing with respect to the control i.e.

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Production and preservation desirable sets

Definition

A desirable set $\mathbb{D}$ is said to be a production desirable set if $\mathbb{D}$ is increasing w.r.t. both the state and to the control, that is

$$\forall \lambda, \lambda' \in \mathbb{U}, \; N, N' \in \mathbb{X} \text{ s.t. } N' \geq N, \; \lambda' \geq \lambda$$
$$\text{if } (N, \lambda) \in \mathbb{D} \text{ then } (N', \lambda') \in \mathbb{D}.$$

Example

$$\mathbb{D}_{\text{yield}} = \{(N, \lambda) \mid Y(N, \lambda) \geq y_{\text{min}}\},$$

where $Y : \mathbb{X} \times \mathbb{U} \longrightarrow R$ is increasing w.r.t. both variables (state and control).

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Example

$$\mathbb{D}_{\text{protect}} = \{(N, \lambda) \in \mathbb{X} \times \mathbb{U} \mid D(N, \lambda) \geq d_\flat\},$$

where $D : \mathbb{X} \times \mathbb{U} \longrightarrow R$ is increasing w.r.t. the state but decreasing w.r.t. the control.

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Viability kernels estimates

Assume that $\lambda_\flat \leq \lambda \leq \lambda_\sharp$ for all $\lambda \in \mathbb{U}$.

Proposition

Suppose that g is an increasing-decreasing dynamics. Then:

- *If $\mathbb{D}$ is a production desirable set, then*

$$\bigcap_{t\geq 0}\{N \in \mathbb{X} \,:\, ((g_\sharp)^t(N), \lambda_\flat) \in \mathbb{D}\} \subseteq$$

$$\mathbb{V}(g, \mathbb{D}) \subseteq \bigcap_{t\geq 0}\{N \in \mathbb{X} \,:\, ((g_\flat)^t(N), \lambda_\sharp) \in \mathbb{D}\}$$

where $g_\flat(\cdot) = g(\cdot, \lambda_\flat)$ and $g_\sharp(\cdot) = g(\cdot, \lambda_\sharp)$

- *If $\mathbb{D}$ is a preservation desirable set, then*

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$$\mathbb{V}(g, \mathbb{D}) \subseteq \bigcap_{t \geq 0} \{N \in \mathbb{X} \, : \, ((g_\flat)^t(N), \lambda_\sharp) \in \mathbb{D}\}$$

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Viability kernels estimates

Assume that $\lambda_\flat \leq \lambda \leq \lambda_\sharp$ for all $\lambda \in \mathbb{U}$.

Proposition

Suppose that g is an increasing-decreasing dynamics. Then:

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Maximum sustainable yield

Assume the existence of a steady state $\overline{N}(\lambda)$ for the dynamics $N \mapsto g(N, \lambda)$, for all $\lambda \in [\lambda_\flat, \lambda_\sharp]$.
Given a yield function $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \boldsymbol{R}$, we define the *maximum sustainable yield* by

$$MSY = \sup_{\lambda \in [\lambda_\flat, \lambda_\sharp]} Y(\overline{N}(\lambda), \lambda) \ .$$

Maximum sustainable yield

Consider the production desirable set $\mathbb{D}_{\text{yield}}$ given by

$$\mathbb{D}_{\text{yield}} = \{(N, \lambda) \mid Y(N, \lambda) \geq y_{\text{min}}\} .$$

Proposition

Suppose that the yield function $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \boldsymbol{R}$ is increasing with respect both to the state and to the control. Then,

- $$MSY \geq y_{\text{min}} \Rightarrow \mathbb{V}(g, \mathbb{D}_{\text{yield}}) \neq \emptyset .$$
- *If the steady state $\overline{N}(\lambda_\flat)$ is globally attractive for the dynamics $g_\flat$, we have*

$$\mathbb{V}(g, \mathbb{D}_{\text{yield}}) \neq \emptyset \Rightarrow Y\big(\overline{N}(\lambda_\flat), \lambda_\sharp\big) \geq y_{\text{min}} .$$

Minimal viable feedback

Define $\mathbb{X}_1 \subset \mathbb{X}$ as those states N such that $SSB(N) \geq B_{\text{lim}}$ and there exists $\lambda^\star(N) \in [\lambda_\flat, \lambda_\sharp]$ satisfying $Y(N, \lambda(N)) = y_{\text{min}}$.

Proposition

$N \in \mathbb{V}(g, \mathbb{D}_{\text{yield}})$, with

$$\mathbb{V}(g, \mathbb{D}_{\text{yield}}) = \{(N, \lambda) \;:\; Y(N, \lambda) \geq y_{\text{min}}, \quad SSB(N) \geq B_{\text{lim}}\},$$

if and only if the trajectory

$$N(t_0) = N\,, \quad N(t+1) = g\big(N(t), \lambda^\star(N(t))\big)\,, \quad t = t_0, t_0+1, \ldots$$

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Outline

Age structured model

- the state : $N = (N_a)_{a=1,\dots,A} \in \mathbb{R}_+^A$, the *abundances* at age
- the control : λ the *fishing effort multiplier*
- the dynamics : $N(t+1) = g(N(t), \lambda(t))$ given by

$$\begin{cases} g_1(N,\lambda) &= \varphi(SSB(N)) \,, \\ g_a(N,\lambda) &= e^{-(M_{a-1}+\lambda F_{a-1})} N_{a-1} \,, \quad a = 2,\dots,A-1 \,, \\ g_A(N,\lambda) &= e^{-(M_{A-1}+\lambda F_{A-1})} N_{A-1} + e^{-(M_A+\lambda F_A)} N_A \,. \end{cases}$$

where

- the *spawning stock biomass* SSB is defined by

$$SSB(N) := \sum_{a=1}^{A} \gamma_a w_a N_a$$

- the function φ describes the *stock-recruitment relationship*
- M_a is the natural *mortality rate* of individuals of age a
- F_a is the mortality rate of individuals of age a due to harvesting

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The harvest term

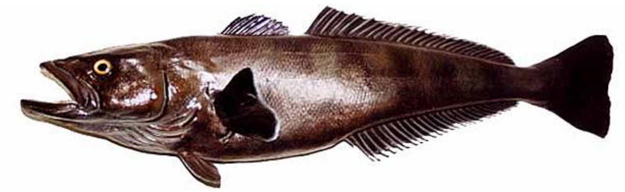
The exploitation is described by catch-at-age C_a and yield Y, both defined for a given vector of abundance N and a given control λ :

$$C_a(N,\lambda) = \frac{\lambda F_a}{\lambda F_a + M_a}\left(1 - e^{-(M_a+\lambda F_a)}\right) N_a$$

The production in term of biomass is :

$$Y(N,\lambda) = \sum_{a=1}^{A} w_a\, C_a(N,\lambda)$$

The Patagonian toothfish (Légine australe)[1]



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  Patagonian toothfish $A = 36$
- Fishing effort multiplier (control): $\lambda \in \mathbb{U} = [\lambda^\flat, \lambda^\sharp]$
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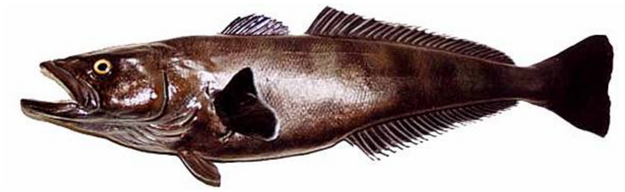

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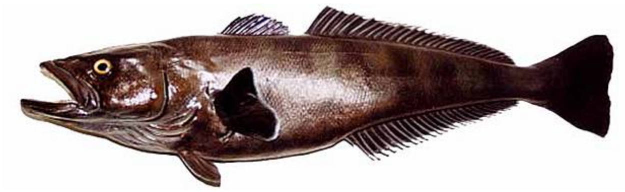

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Some questions about sustainability of landings

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1,..,A}$ (initial conditions) for which one can always harvest at least that quantity?
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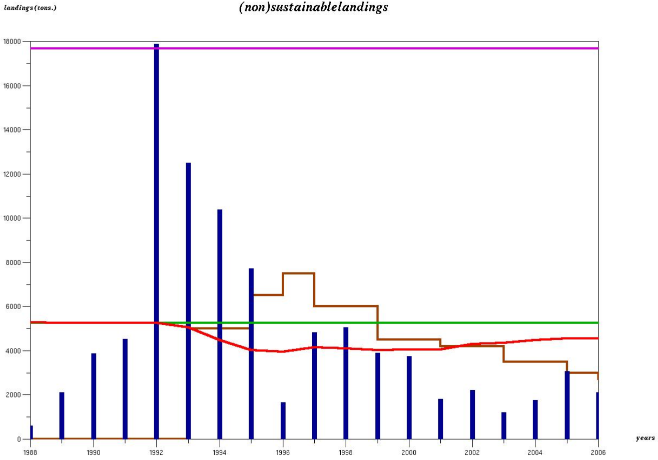
Results

■	=	landings
red line	=	maximum sustainable yield (considering the abundance)
dark red line	=	quota of the government regulatory Chilean agency (SUBPESCA)
magenta line	=	non sustainable level (independently of the abundance)
green line	=	maximum sustainable yield in the equilibrium



Open questions

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