Viable harvest of monotone bioeconomics models

Preservation and production issues

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Outline

The core model for fishery management decisions
  The model
  Some questions about sustainability of landings

Discrete time viability issues
  Monotonicity properties

Viability kernel properties
  Maximum sustainable yield
  Minimal viable feedback

The Patagonian toothfish (Légine australe)
  Questions & Answers
Outline

The core model for fishery management decisions

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The Patagonian toothfish (Légine australè)
Questions & Answers
Age structured model

- **the state**: \( N = (N_a)_{a=1,\ldots,A} \in \mathbb{R}^A \), the *abundances* at age
- **the control**: \( \lambda \) the *fishing effort multiplier*
- **the dynamics**: \( N(t+1) = g(N(t), \lambda(t)) \) given by

\[
\begin{align*}
    g_1(N, \lambda) &= \varphi(SSB(N)), \\
    g_a(N, \lambda) &= e^{-(M_a-1+\lambda F_{a-1})}N_{a-1}, \quad a = 2, \ldots, A - 1, \\
    g_A(N, \lambda) &= e^{-(M_A-1+\lambda F_{A-1})}N_{A-1} + e^{-(M_A+\lambda F_A)}N_A.
\end{align*}
\]

where
- **the spawning stock biomass** \( SSB \) is defined by

\[
SSB(N) = \sum_{a=1}^{A} \gamma_a w_a N_a
\]
- **the function** \( \varphi \) describes the *stock-recruitment relationship*
- **\( M_a \)** is the *natural mortality rate* of individuals of age \( a \)
- **\( F_a \)** is the *mortality rate* of individuals of age \( a \) due to harvesting
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Harvested fish population age structured model

Example

Typical stock-recruitment relationship:

- **Constant:** $\varphi(B) = R$.
- **Linear:** $\varphi(B) = RB$.
- **Beverton-Holt:** $\varphi(B) = \frac{B}{\alpha + \beta B}$.
- **Ricker:** $\varphi(B) = \alpha B e^{-\beta B}$. 
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The harvest term

The exploitation is described by catch-at-age $C_a$ and yield $Y$, both defined for a given vector of abundance $N$ and a given control $\lambda$ :

$$C_a(N, \lambda) = \frac{\lambda F_a}{\lambda F_a + M_a} \left(1 - e^{-(M_a + \lambda F_a)}\right) N_a$$

The production in term of biomass is :

$$Y(N, \lambda) = \sum_{a=1}^{A} w_a \ C_a(N, \lambda)$$
Questions

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1, \ldots, A}$ (initial conditions) for which one can always harvest at least that quantity?
- What levels of catch (landings) are non sustainable?
- Given an abundance at age $N = (N_a)_{a=1, \ldots, A}$ What is the maximum sustainable yield starting from $N$ respecting preservation constraints?
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Discrete time control system

Let us consider a nonlinear control system described in discrete time by the difference equation

\[
\begin{align*}
N(t+1) &= g(N(t), \lambda(t)), \quad \forall t \in \mathbb{N}, \\
N_0 & \text{ given,}
\end{align*}
\]

where

- The state variable \( N(t) \) belongs to the state space \( X \subseteq \mathbb{R}^n \).
- The control variable \( \lambda(t) \) is an element of the control set \( U \subseteq \mathbb{R}^m \).
- The dynamics \( g \) maps \( X \times U \) into \( X \).
Discrete time control system

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Desirable configurations

A decision maker describes desirable configurations of the system through a set $\mathcal{D} \subset X \times U$ termed the desirable set

$$(N(t), \lambda(t)) \in \mathcal{D}, \quad \forall t \in \mathbb{N},$$

where $\mathcal{D}$ includes both system states and controls constraints.

Example

- $\mathcal{D}_{\text{protect}} := \{(N, \lambda) : N \geq \bar{N}\}$
- $\mathcal{D}_{\text{yield}} := \{(N, \lambda) : Y(N, \lambda) \geq y_{\text{min}}, \quad S\mathcal{S}B(N) \geq B_{\text{lim}}\}$
- $\mathcal{D}_{\text{ICES}} := \{(N, \lambda) : S\mathcal{S}B(N) \geq B_{\text{lim}}, \quad F(\lambda) \leq F_{\text{lim}}\}$
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Viability domains and viability kernel

Definition

- \( \forall \subset X \) is a **Viability Domain** if for all \( N \in \forall \) there exists \( \lambda \in U \) such that \( (N, \lambda) \in D \) and \( g(N, \lambda) \in \forall \).

- **Viability kernel**

\[
\forall(g, D) = \left\{ N_0 \in X : \text{there exist } \lambda(0), \lambda(1), \lambda(2), ... \text{such that } N(0) = N_0, N(1), N(2), ... \text{such that } N(0) = N_0, N(t + 1) = g(N(t), \lambda(t)) \text{ and } (N(t), \lambda(t)) \in D \right\}
\]

Goals

- Determine or approximate the viability kernel \( \forall(g, D) \) for a given dynamics \( g \) and a given desirable set \( D \).
- Determine when a given set \( \forall \) is a viability domain.
Viability domains and viability kernel

Definition

- \( \mathcal{V} \subset \mathbb{X} \) is a Viability Domain if for all \( N \in \mathcal{V} \) there exists \( \lambda \in \mathbb{U} \) such that \( (N, \lambda) \in \mathbb{D} \) and \( g(N, \lambda) \in \mathcal{V} \).

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\mathcal{V}(g, \mathbb{D}) = \left\{ N_0 \in \mathbb{X} : \text{there exist } \lambda(0), \lambda(1), \lambda(2), \ldots \text{such that } N(0) = N_0, N(t + 1) = g(N(t), \lambda(t)) \text{ and } (N(t), \lambda(t)) \in \mathbb{D} \right\}
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$$\mathcal{V}(g, \mathbb{D}) = \begin{cases} N_0 \in \mathbb{X} : \text{there exist } \lambda(0), \lambda(1), \lambda(2), \ldots \\ N(0), N(1), N(2), \ldots \text{such that } N(0) = N_0 \\ N(t + 1) = g(N(t), \lambda(t)) \text{ and} \\ (N(t), \lambda(t)) \in \mathbb{D} \end{cases}$$

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Viability kernel

$$\forall(g, D) = \begin{cases} N_0 \in X : \text{there exist } \lambda(0), \lambda(1), \lambda(2), \ldots, N(0), N(1), N(2), \ldots \text{such that } N(0) = N_0, N(t + 1) = g(N(t), \lambda(t)) \text{ and } (N(t), \lambda(t)) \in D \end{cases}$$

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Monotonicity properties on the dynamics

Definition

We say that the dynamics $g : \mathbb{X} \times \mathbb{U} \to \mathbb{X}$ is a monotone bioeconomic dynamics if $g$ is increasing with respect to the state i.e.

$$\forall (N, N', \lambda) \in \mathbb{X} \times \mathbb{X} \times \mathbb{U}, \quad N' \geq N \Rightarrow g(N', \lambda) \geq g(N, \lambda),$$

and is decreasing with respect to the control i.e.

$$\forall (N, \lambda, \lambda') \in \mathbb{X} \times \mathbb{U} \times \mathbb{U}, \quad \lambda' \geq \lambda \Rightarrow g(N, \lambda') \leq g(N, \lambda).$$
Monotonicity properties on the dynamics

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$$\forall (N, \lambda, \lambda') \in \mathbb{X} \times U \times U, \quad \lambda' \geq \lambda \Rightarrow g(N, \lambda') \leq g(N, \lambda).$$
Production and preservation desirable sets

Definition

A desirable set $\mathbb{D}$ is said to be a production desirable set if $\mathbb{D}$ is increasing w.r.t. both the state and to the control, that is

$$\forall \lambda, \lambda' \in \mathbb{U}, \ N, N' \in \mathbb{X} \ s.t. \ N' \geq N, \ \lambda' \geq \lambda$$

if $(N, \lambda) \in \mathbb{D}$ then $(N', \lambda') \in \mathbb{D}$.

Example

$$\mathbb{D}_{\text{yield}} = \{(N, \lambda) \mid Y(N, \lambda) \geq y_{\text{min}}\},$$

where $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{R}$ is increasing w.r.t. both variables (state and control).
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if $(N, \lambda) \in \mathbb{D}$ then $(N', \lambda') \in \mathbb{D}$.

Example

$$\mathbb{D}_{\text{protect}} = \{(N, \lambda) \in \mathbb{X} \times \mathbb{U} \mid D(N, \lambda) \geq d_b\},$$

where $D : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$ is increasing w.r.t. the state but decreasing w.r.t. the control.
Production and preservation desirable sets

Definition

A desirable set $D$ is said to be a preservation desirable set if $D$ is increasing w.r.t. the state and decreasing w.r.t. the control:

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Viability kernels estimates

Assume that $\lambda_b \leq \lambda \leq \lambda_\#$ for all $\lambda \in \mathbb{U}$.

Proposition

Suppose that $g$ is an increasing-decreasing dynamics. Then:

- If $\mathcal{D}$ is a production desirable set, then

  $$\bigcap_{t \geq 0} \{ N \in X : ((g_\#)^t(N), \lambda_b) \in \mathcal{D} \} \subseteq \mathbb{V}(g, \mathcal{D}) \subseteq \bigcap_{t \geq 0} \{ N \in X : ((g_b)^t(N), \lambda_\#) \in \mathcal{D} \}$$

  where $g_b(\cdot) = g(\cdot, \lambda_b)$ and $g_\#(\cdot) = g(\cdot, \lambda_\#)$

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Viability kernels estimates

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Maximum sustainable yield

Assume the existence of a steady state $\bar{N}(\lambda)$ for the dynamics $N \mapsto g(N, \lambda)$, for all $\lambda \in [\lambda_b, \lambda^#]$. Given a yield function $Y : \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$, we define the maximum sustainable yield by

$$MSY = \sup_{\lambda \in [\lambda_b, \lambda^#]} Y(\bar{N}(\lambda), \lambda).$$
Maximum sustainable yield

Consider the production desirable set $\mathbb{D}_{\text{yield}}$ given by

$$\mathbb{D}_{\text{yield}} = \{(N, \lambda) \mid Y(N, \lambda) \geq y_{\text{min}}\}.$$ 

**Proposition**

Suppose that the yield function $Y : \mathbb{X} \times \mathbb{U} \longrightarrow \mathbb{R}$ is increasing with respect both to the state and to the control. Then,

- $MSY \geq y_{\text{min}} \Rightarrow \mathbb{V}(g, \mathbb{D}_{\text{yield}}) \neq \emptyset$.

- If the steady state $\bar{N}(\lambda_b)$ is globally attractive for the dynamics $g_b$, we have

$$\mathbb{V}(g, \mathbb{D}_{\text{yield}}) \neq \emptyset \Rightarrow Y(\bar{N}(\lambda_b), \lambda^\flat) \geq y_{\text{min}}.$$
Minimal viable feedback

Define $X_1 \subset X$ as those states $N$ such that $SSB(N) \geq B_{\text{lim}}$ and there exists $\lambda^*(N) \in [\lambda_b, \lambda^*]$ satisfying $Y(N, \lambda(N)) = y_{\text{min}}$.

Proposition

$N \in V(g, D_{\text{yield}})$, with

$$V(g, D_{\text{yield}}) = \{ (N, \lambda) : Y(N, \lambda) \geq y_{\text{min}}, \quad SSB(N) \geq B_{\text{lim}} \},$$

if and only if the trajectory

$$N(t_0) = N, \quad N(t+1) = g(N(t), \lambda^*(N(t))) , \quad t = t_0, t_0+1, \ldots$$

is well defined, namely $N(t) \in X_1$. 
**Minimal viable feedback**

Define $X_1 \subset X$ as those states $N$ such that $\mathcal{SB}(N) \geq B_{\text{lim}}$ and there exists $\lambda^*(N) \in [\lambda_b, \lambda^\#]$ satisfying $Y(N, \lambda(N)) = y_{\text{min}}$.

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Outline

The core model for fishery management decisions
  The model
  Some questions about sustainability of landings

Discrete time viability issues
  Monotonicity properties

Viability kernel properties
  Maximum sustainable yield
  Minimal viable feedback

The Patagonian toothfish (Légine australe)
  Questions & Answers
Age structured model

- **the state**: \( N = (N_a)_{a=1,...,A} \in \mathbb{R}_+^A \), the abundances at age
- **the control**: \( \lambda \) the *fishing effort multiplier*
- **the dynamics**: \( N(t+1) = g(N(t), \lambda(t)) \) given by

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\begin{align*}
g_1(N, \lambda) &= \varphi(\text{SSB}(N)), \\
g_a(N, \lambda) &= e^{-(M_{a-1} + \lambda F_{a-1})}N_{a-1}, \quad a = 2, \ldots, A - 1, \\
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where

- the *spawning stock biomass* \( \text{SSB} \) is defined by

\[
\text{SSB}(N) := \sum_{a=1}^{A} \gamma_a w_a N_a
\]

- the function \( \varphi \) describes the *stock-recruitment relationship*
- \( M_a \) is the natural *mortality rate* of individuals of age \( a \)
- \( F_a \) is the mortality rate of individuals of age \( a \) due to harvesting
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The harvest term

The exploitation is described by catch-at-age $C_a$ and yield $Y$, both defined for a given vector of abundance $N$ and a given control $\lambda$:

$$C_a(N, \lambda) = \frac{\lambda F_a}{\lambda F_a + M_a} \left(1 - e^{-(M_a + \lambda F_a)}\right) N_a$$

The production in term of biomass is:

$$Y(N, \lambda) = \sum_{a=1}^{A} w_a C_a(N, \lambda)$$
The Patagonian toothfish (Légine australe)$^1$

- Abundance at age (state): $N = (N_a)_{a=1,...,A}$
  - Patagonian toothfish $A = 36$
- Fishing effort multiplier (control): $\lambda \in \mathbb{U} = [\lambda^b, \lambda^#]$
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- Stock-recruitment relationship $\varphi$
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$^1$Data: CEPES, SUBPESCA, Chile
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Some questions about sustainability of landings

- Given a desirable level of landings (tons), what are the vectors of abundances $N = (N_a)_{a=1,\ldots,A}$ (initial conditions) for which one can always harvest at least that quantity?
- What levels of catch (landings) are non sustainable?
- Given an abundance at age $N = (N_a)_{a=1,\ldots,A}$ What is the maximum sustainable yield starting from $N$ and satisfying preservation constraints?
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Results

- landings
- maximum sustainable yield (considering the abundance)
- quota of the government regulatory Chilean agency (SUBPESCA)
- non sustainable level (independently of the abundance)
- maximum sustainable yield in the equilibrium

![Graph showing landings and sustainable landings over years]
Open questions

- To consider a vector control $\lambda = (\lambda_1, \ldots, \lambda_p)$
- Interaction between species:
  - Technical interactions
  - Biological interactions $\Rightarrow$ to consider a non-monotone dynamics $g$
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