Multivalued complementarity problems with asymptotically bounded multifunctions

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Contents

Motivation

Consider

$$\min_{x \geq 0} h(x), \tag{1}$$

h differentiable on $\mathbb{R}^n_+$. The KKT optimality condition says: if $\bar{x} \geq 0$ is a solution to (1), then there exists $\lambda \geq 0$ such that

$$\nabla h(\bar{x}) - \lambda = 0$$

$$\langle \lambda, \bar{x} \rangle = 0.$$

Moreover, it is sufficient when h is pseudoconvex.

By replacing $\nabla h(x)$ by $F(x)$, the KKT condition leads

$$\bar{x} \geq 0 : F(\bar{x}) \geq 0, \ \langle F(\bar{x}), \bar{x} \rangle = 0.$$

Formulation

Let $q \in \mathbb{R}^n$, $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be given:

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in F(\bar{x}) :\ \bar{y} + q \geq 0,\ \langle \bar{y} + q, \bar{x} \rangle = 0 \quad (MCP(q, F))$$

$$\text{find } \bar{x} \geq 0,\ \bar{y} \in F(\bar{x}) :\ \langle \bar{y} + q, x - \bar{x} \rangle \geq 0 \ \ \forall\, x \geq 0 \quad (VIP(q, F))$$

$\mathcal{S}(q, F)$: set of solutions to $(MCP(q, F))$.

- $F(x) \neq \emptyset \ \forall\, x \geq 0$;
- $F(x)$ is convex, compact $\forall\, x \geq 0$;
- F is upper semicontinuous on $\mathbb{R}^n_+$.

Karamardian S. 1972, 1976; Moré J.J 1974; García C.B. 1973; Saigal R. 1976; Isac G. 1989,...; Gowda S.M., Pang J.S. 1992; Crouzeix 1997; ....

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Example 1. $M \in \mathbb{R}^{n\times n}$, $A, B \in \mathbb{R}^{m\times n}$, $q \in \mathbb{R}^n$. Set

$$F(x) = \{Mu : \ Au + Bx \leq 0\}, \ x \geq 0.$$

The problem is to find $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, $u \in \mathbb{R}^n$ such that

$$x \geq 0, \ y = Mu + q \geq 0, \ \langle Mu + q, x\rangle = 0, Au + Bx \leq 0.$$

Example 2. $F(x) = Mx + \partial h(x)$, $M \in \mathbb{R}^{n\times n}$,

$$h(x) = \sup_{u\in C}\langle x, u\rangle,$$

$C \neq \emptyset$, convex, compact. If M is symmetric, the MCP is the stationary point problem of the following minimax problem

$$\min_{x\geq 0}\sup_{u\in C}\left\{\frac{1}{2}\langle x, Mx\rangle + \langle x, q+u\rangle\right\}$$

Set $\mathbb{R}_{++} = \;]0, +\infty[$. Let us consider

$$\mathcal{C} \doteq \left\{c : \mathbb{R}_{++} \to \mathbb{R}_{++} : \lim_{t\to+\infty} c(t) = +\infty\right\}$$

$$\mathcal{C}_0 \doteq \left\{c \in \mathcal{C} : \lim_{t\to+\infty} \frac{c(\lambda t)}{c(t)} \in \mathbb{R} \text{ for all } \lambda > 0\right\}.$$

For $c \in \mathcal{C}_0$, set

$$c^\infty(\lambda) \doteq \lim_{t\to+\infty} \frac{c(\lambda t)}{c(t)}; c^\infty(1) = 1; c^\infty(ts) = c^\infty(t)c^\infty(s),\; t, s > 0.$$

Examples

$$c_1(t) = t^p (p > 0),\; c_1^\infty(t) = t^p,$$

$$c_2(t) = t^p \ln(t^\gamma + 1), c_2(t) = \frac{t^p}{\ln(t^\gamma + 1)}, (p > 0, \gamma \geq 1), c_2^\infty(t) = t^p.$$

Asymptotic bounded multifunctions:

Given a sequence of multifunctions $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$, $k \in \mathbb{N}$ and $c \in \mathcal{C}$, the c-asymptotic multifunction associated to $\{F^k\}$ is defined by

$$\limsup_k{}^{\infty,c} F^k(v) \doteq \Big\{ w \in \mathbb{R}^n : \ \lambda_{k_m} \uparrow +\infty, \ x^{k_m} \geq 0, \ \frac{x^{k_m}}{\lambda_{k_m}} \to v, \\ y^{k_m} \in F^{k_m}(x^{k_m}), \ \frac{y^{k_m}}{c(\lambda_{k_m})} \to w \Big\}.$$

Indeed,

$$\text{gph}(\limsup_k{}^{\infty,c} F^k) = \limsup_k{}^{\infty,c}(\text{gph } F^k),$$

where,

$$\limsup_{k}{}^{\infty,c}(\text{gph } F^k) \doteq \Big\{(v,w)\in \mathbb{R}^n_+ \times \mathbb{R}^n : \ \lambda_{k_m}\uparrow +\infty,$$

$$(x^{k_m}, y^{k_m}) \in \text{gph } F^{k_m},\ (\frac{x^{k_m}}{\lambda_{k_m}}, \frac{y^{k_m}}{c(\lambda_{k_m})}) \to (v,w)\Big\}.$$

c-asymptotically bounded property

The property of *c-asymptotically bounded at v* (c-AB) means: for every $k_m \to +\infty$, every $\lambda_{k_m} \uparrow +\infty$, as $m \to +\infty$, every $x^{k_m} \geq 0$, every $y^{k_m} \in F^{k_m}(x^{k_m})$, such that $\{x^{k_m}/\lambda_{k_m}\}$ converges to v, the sequence $\{y^{k_m}/c(\lambda_{k_m})\}$ is bounded.

When $F^k = F$, $c(t) = t^\gamma$, $\gamma > 0$ it was discussed by Gowda-Pang, 92.

Given $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ and $c \in \mathcal{C}$, we set

$$A_F(x) \doteq \sup_{y \in F(x)} |y|.$$

The above property holds iff

$$\limsup_k{}^{\infty} A_{F^k,c}(v) \doteq \sup\left\{ \limsup_{m\to+\infty} \frac{1}{c(\lambda_{k_m})} A_{F^{k_m}}(\lambda_{k_m} x^{k_m}) : \right.$$

$$\left. k_m \uparrow +\infty, \lambda_{k_m} \uparrow +\infty,\ x^{k_m} \to v \right\} < +\infty.$$

As before, in case $F^k = F$ for all k, we set

$$F_c^{\infty}(v) \doteq \limsup_k{}^{\infty,c} F^k(v),\ \ A_{F,c}^{\infty}(v) \doteq \limsup_k{}^{\infty} A_{F^k,c}(v).$$

More precisely, one has the following lemma.

Lemma: Flores-Bazán, 2007

Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ with nonempty compact values, $c \in \mathcal{C}$ and $v \geq 0$. Then, the following assertions are equivalent:

(a) $\limsup_k^\infty A_{F^k,c}(v) < +\infty$;

(b) $\{F^k\}$ satisfies the c-asymptotically bounded property at v.

Examples: Take $M, M^k \in \mathbb{R}^{n\times n}$ satisfying $M^k \to M$, and $\rho \in \mathbb{R}$:

- $F^k(x) = |x|^\gamma M^k x - \rho x$ $(\gamma \geq 0)$, $c(t) = t^{\gamma+1} = c^\infty(t)$, then

$$\limsup_k{}^\infty A_{F^k,c}(v) = |v|^\gamma |Mv|, \quad \limsup_k{}^{\infty,c} F^k(v) = |v|^\gamma Mv.$$

- $F^k(x) = \ln(|x|^\gamma + 1)M^k x - \rho x$ $(\gamma \geq 1)$, $c(t) = t\ln(t^\gamma + 1)$, then $c^\infty(t) = t$.

$$\limsup_k{}^\infty A_{F^k,c}(v) = |Mv|, \quad \limsup_k{}^{\infty,c} F^k(v) = Mv.$$

- Take $\gamma > 0$ and $\emptyset \neq C^k \subseteq \mathbb{R}^n$, closed, converging (in the sense of Painlevé-Kuratowski) to a closed $\emptyset \neq C \subseteq \mathbb{R}^n$.

$$F^k(x) = |x|^\gamma C^k, \quad \limsup_k{}^{\infty,c} F^k(v) = |v|^\gamma C.$$

Examples: Continued ...

- Take $F(x) = \ln(|x|^\gamma + 1)H(x)$ or $F(x) = \frac{1}{\ln(|x|^\gamma+1)}H(x)$ ($\gamma \geq 1$), with H being positive homogeneous of degree $p > 0$, that is $H(tx) = t^p H(x)\ \forall\ t > 0,\ x \geq 0$. Consider $c(t) = t^p \ln(t^\gamma + 1)$ in the first case, and $c(t) = \frac{t^p}{\ln(t^\gamma+1)}$ in the second case. Here $c^\infty(t) = t^p$.
- $F(x) = M_1 x + \ln(|x| + 1)M_2 x$, where $M_i \in \mathbb{R}^{n\times n}$ Here we use $c(t) = t\ln(t + 1)$. Here $c^\infty(t) = t$.
- Take $F(x) = (\langle M_1 x, x\rangle, \ln(|x| + 1)\langle M_2 x, x\rangle) \in \mathbb{R}^2$, $x \geq 0$. Here $c(t) = t^2 \ln(t + 1)$ is useful.

Theorem (Flores-Bazán, 2007): Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ with nonempty compact values $c \in \mathcal{C}$. The following assertions hold.

(*a*) If $\limsup_k^{\infty} A_{F^k,c}(v) < +\infty$ then $\limsup_k{}^{\infty,c} F^k(v) \neq \emptyset$,

$$\limsup_k{}^{\infty} A_{F^k,c}(v) = \sup\left\{|w| : \; w \in \limsup_k{}^{\infty,c} F^k(v)\right\}.$$

(*b*) If $c \in \mathcal{C}_0$ and $c^\infty(\lambda) > 0 \; \forall \; \lambda > 0$, then

$$\limsup_k{}^{\infty,c} F^k \; \textit{is } c^\infty\textit{-homogeneous}$$

(i.e., $G(tx) = c^\infty(t)G(x), \; \forall \; t > 0, \; x \geq 0$).

Proposition: $F^k, G^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be with nonempty values for $k \in \mathbb{N}$; $c \in \mathcal{C}$ $v \geq 0$.

$$\text{If } \limsup_k{}^{\infty} A_{G^k,c}(v) = 0 \text{ then}$$

$$\limsup_k{}^{\infty,c} H^k(v) = \limsup_k{}^{\infty,c} F^k(v),$$

where $H^k = F^k + G^k$.

Take $G^k(x) = \partial\sigma_{C^k}(x)$, $k \in \mathbb{N}$, where C^k is any nonempty compact convex set in $\mathbb{R}^n$ which converges (in the sense of Painlevé-Kuratowski) to the nonempty compact convex set C, is a typical example satisfying the above condition for every $c \in \mathcal{C}$. In fact, due to the convergence of C^k, these sets remain in a fixed bounded set.

Theorem [Flores-Bazán, 2007]

Let $F^k, F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be multifunctions with nonempty values for $k \in \mathbb{N}$; $c \in \mathcal{C}$, $v \geq 0$. Assume that $F^k \xrightarrow{g} F$, then

$$F_c^\infty(v) \subseteq \limsup_k{}^{\infty,c} F^k(v).$$

If, in addition, each F^k is c-subhomogeneous, then

$$\limsup_k{}^{\infty,c} F^k(v) \subseteq F(v).$$

$F^k(tx) \subseteq c(t)F^k(x)\ \forall\ t > 0, \forall\ x \geq 0.$

Example: Take $\{H^k\} \subseteq \mathcal{X}$ which is asymptotically equi-osc and

converging pointwise to $H \in \mathcal{X}$ (See Sections E and F of Chap. 5 in [RW]). By Theorem 5.40 in [RW], $H^k \xrightarrow{g} H$. Assume for some $p > 0$, $H^k(tx) = t^p H^k(x) \;\; \forall\, t > 0, \; \forall\, x \geq 0$.
Then $H(tx) = t^p H(x) \;\; \forall\, t > 0, \; \forall\, x \geq 0$. By setting $F^k(x) = \ln(|x| + 1)H^k(x)$, as above, $F^k \xrightarrow{g} F$ with $F(x) = \ln(|x| + 1)H(x)$, and

$$\limsup_k{}^{\infty,c} F^k(v) = F_c^{\infty}(v) = H(v), \quad c(t) = t^p \ln(t+1).$$

A model is given by $H^k(x) = \{M^k y : \; Ax + By \leq 0\}$, where $M^k \in \mathbb{R}^{n\times n}$, $A, B \in \mathbb{R}^{m\times n}$, $p = 1$.

Some notations:

- given $J \subseteq I \doteq \{1, \ldots, n\}$ and $d > 0$ (component-wise); set $\Delta_J = \Delta_J(d) \doteq \mathrm{co}\{\frac{1}{d_i} e^i : \ i \in J\}$, where e^i is the i-th column of the identity matrix in $\mathbb{R}^{n \times n}$. However, sometimes we will omit the dependence on d when no confusion arises; denote $\Delta_d \doteq \{x \geq 0 : \ \langle d, x \rangle = 1\} = \Delta_I$;
- given $x \in \mathbb{R}^n$, we set $\mathrm{supp}\{x\} \doteq \{i \in I : \ x_i \neq 0\}$;
- given $k \in \mathbb{N}$, $d > 0$ (component-wise), we set

$$D_k \doteq \{x \geq 0 : \ \langle d, x \rangle \leq \sigma_k\}.$$

The Basic Lemma [FB-López, 2005; FB, 2007]

$F^k, G^k \in \mathcal{X}$, and $\{(x^k, y^k, r^k)\}$ be a sequence of solutions to

$$\text{find } x^k \in D_k : \ \ y^k \in F^k(x^k),\ r^k \in G^k(x^k),$$

$$\langle y^k + r^k + q^k, x - x^k \rangle \geq 0 \ \forall\, x \in D_k. \qquad (VI_k)$$

such that $\langle d, x^k \rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Then, there exist subsequences $\{\sigma_{k_m}\}$ and $\{(x^{k_m}, y^{k_m}, r^{k_m})\}$, numbers k_0, $m_0 \in \mathbb{N}$, and an index set $\emptyset \neq J_v \subseteq I$ such that

(*a*) for all $k \geq k_0$, $x^k - \frac{\sigma_k}{2} v \geq 0$ and $0 < \langle d, x^k - \frac{\sigma_k}{2} v \rangle < \langle d, x^k \rangle$;

(*b*) for all $m \geq m_0$, $\frac{1}{\sigma_{k_m}} x^{k_m} \in \mathrm{ri}(\Delta_{J_v})$, thus $\mathrm{supp}\{x^{k_m}\} = J_v$ (hence $\mathrm{supp}\{v\} \subseteq J_v$);

The Basic Lemma: continued ...

(*c*) for all $m \geq m_0$, $z \in \Delta_{J_v}$: $\langle y^{k_m} + r^{k_m} + q^{k_m}, \sigma_{k_m} z - x^{k_m} \rangle = 0$. Moreover, for a given $c \in \mathcal{C}$,

(*d*) if

$$\limsup_k{}^{\infty} A_{F^k,c}(v) < +\infty \;\; \textit{and} \; \limsup_k{}^{\infty} A_{G^k,c}(v) = 0,$$

then the subsequences $\{y^{k_m}\}$, σ_{k_m} may be chosen in such a way that there is a vector w such that $\langle w, v \rangle \leq 0$, $\frac{1}{c(\sigma_{k_m})} y^{k_m} \to w \in \limsup_k^{\infty,c} F^k(v)$, $\langle w, y \rangle \geq \langle d, y \rangle \langle w, v \rangle$ for all $y \geq 0$, and $\langle w, z \rangle = \langle w, v \rangle$ for all $z \in \Delta_{J_v}$.

Remark

In the previous lemma we actually get $\langle v, d\rangle = 1$. Additionally, by choosing $y = e^i$, $i = 1, \dots, n$, in (d), and setting $z \doteq w - \langle w, v\rangle d \geq 0$, we obtain $\langle z, v\rangle = 0$, $w \in \limsup_k^{\infty,c} F^k(v)$. Therefore, if $\tau \doteq -\langle w, v\rangle \geq 0$, then

$$J_v \neq \emptyset,\ v \in \Delta_{J_v},\ v \neq 0,\ \ v \in \mathcal{S}(\tau d, \limsup_k{}^{\infty,c} F^k).$$

(straightforward)Theorem: Let $d > 0$, $c \in \mathcal{C}$ such that

$c^\infty(\lambda) > 0\ \forall\ \lambda > 0$; let $F \in \mathcal{X}$. If $A^\infty_{F,c}(v) < +\infty\ \ \forall\ v \in \Delta_d$ and F^∞_c is strictly copositive, then $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ satisfying $A^\infty_{G,c}(v) = 0\ \ \forall\ v \in \Delta_d$.

Proposition: Let $d > 0$, $q \in \mathbb{R}^n_+$, $c \in \mathcal{C}$, $F, G \in \mathcal{X}$

$$A^{\infty}_{F,c}(v) < +\infty \text{ and } A^{\infty}_{G,c}(v) = 0 \;\; \forall\, v \in \Delta_d.$$

Then,

$$(\mathcal{S}(q, F + G))^{\infty} \subseteq \mathcal{S}(0, F^{\infty}_c).$$

General existence theorem [FB, 2007]

Let $d > 0$, $\sigma_k \uparrow +\infty$ and $q \in \mathbb{R}^n$; let $F \in \mathcal{X}$, and $\{(x^k, y^k)\}$ be a sequence of solutions to

$$\text{find } x^k \in D_k : \quad y^k \in F(x^k), \;\; \langle y^k + q, x - x^k \rangle \geq 0 \;\forall\, x \in D_k. \;\; (VI)$$

such that $\langle d, x^k \rangle = \sigma_k$ and $\frac{x^k}{\sigma_k} \to v$ as $k \to +\infty$. Then, the following assertions are equivalent:

- (a) $\exists\; m_0$ and $\{x^{k_m}\}$ such that $\langle y^{k_m} + q, v \rangle \geq 0 \;\forall\; m \geq m_0$;
- (b) $\exists\; m_0$ and $\{x^{k_m}\}$ such that $\forall\; m \geq m_0 \;\exists\; u^{k_m} \geq 0$, $0 < \langle d, u^{k_m} \rangle < \langle d, x^{k_m} \rangle$ and $\langle y^{k_m} + q, u^{k_m} - x^{k_m} \rangle \leq 0$.
- (c) $\exists\, m_0$ and $\{x^{k_m}\}$ such that $x^{k_m} \in \mathcal{S}(q, F) \;\forall\; m \geq m_0$;
- (d) $\exists\; m_0$ and $\{x^{k_m}\}$ such that $\langle y^{k_m} + q, v \rangle = 0 \;\forall\; m \geq m_0$.

Classes of Mappings

Let $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be a multifunction with nonempty values. We say that F is a:

(*i*) copositive mapping if $\langle x, y \rangle \geq 0 \;\; \forall \, (x, y) \in \operatorname{gph} F$;

(*ii*) strictly copositive mapping if $\langle x, y \rangle > 0 \;\; \forall \, (x, y) \in \operatorname{gph} F$, $x \neq 0$;

(*iii*) (assuming $0 \in F(0)$) semimonotone mapping if $\mathcal{S}(p, F) = \{0\} \;\; \forall \, p > 0$; (given $p > 0$) **G**(p)- mapping or shortly $F \in \mathbf{G}(p)$ if $\mathcal{S}(\tau p, F) = \{0\} \;\; \forall \, \tau > 0$.

In the case when F is c-homogeneous with $c(t) = t^\gamma$, $\gamma > 0$, we get

$$F \in \mathbf{G}(p) \Longleftrightarrow \mathcal{S}(p, F) = \{0\},$$

since $v \in \mathcal{S}(\tau p, F) \Longleftrightarrow \tau^{-1/\gamma} v \in \mathcal{S}(p, F)$.

Definition

Given $d >$ and $c \in \mathcal{C}$, we say that $F : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ is *asymptotically well-behaved* **T**-*mapping*,
if $A^\infty_{F,c}(v) < +\infty \; \forall \; v \in \Delta_d$, and for any index set $\alpha \subseteq I$, one has

$$\left.\begin{array}{ll} v \geq 0, \;\; w \geq 0, & w \in F^\infty_c(v), \\ \alpha \neq \emptyset, \;\; v \in \Delta_\alpha, & w_\alpha = 0 \end{array}\right\} \Longrightarrow v \in [F(\mathrm{pos}^+ \Delta_\alpha)]^*.$$

When $c \in \mathcal{C}_0$, F^∞_c is c^∞-homogeneous, so this definition is independent of d.
Any F satisfying $A^\infty_{F,c}(v) < +\infty \; \forall \; v \in \Delta_d$ and $\mathcal{S}(0, F^\infty) = \{0\}$ is T....

The Linear Case: $F(x) = Mx$ [FB-López, 2005]

Given $d >$, we say that M is a **T**-matrix,
if one has

$$\left.\begin{array}{ll} 0 \neq v \geq 0, & Mv \geq 0, \\ (Mv)_\alpha = 0, & \mathrm{supp}\{v\} \subseteq \alpha, \end{array}\right\} \Longrightarrow (M^\top v)_\alpha \geq 0.$$

It generalizes the class of $\#$-matrix introduced in [Gowda-Pang, 1993]. It contains properly the symmetric matrices, the copositives and those satisfying $\mathcal{S}(0, M) = \{0\}$.
We say that M is $\#$-matrix if

$$v \in \mathcal{S}(0, M) \Longrightarrow (M + M^\top)(v) \geq 0.$$

Theorem [FB, 2007]

Let $d > 0$, $c \in \mathcal{C}_0$ such that $0 < c^\infty(t)\ \forall\ t > 0$. Let $F^k : \mathbb{R}^n_+ \hookrightarrow \mathbb{R}^n$ be a sequence asymptotically well-behaved **T**-mapping such that $F_c^\infty \in \mathbf{G}(d)$.

(a) If $q \in [\mathcal{S}(0, F_c^\infty)]^\#$ then $\mathcal{S}(q, F + G) \neq \emptyset$ and compact for all $G \in \mathcal{X}$ copositive and zero-subhomogeneous.

(b) If $q \in [\mathcal{S}(0, F_c^\infty)]^* \setminus [\mathcal{S}(0, F_c^\infty)]^\#$, then $\mathcal{S}(q, F + G) \neq \emptyset$ and compact for all $G \in \mathcal{X}$ strictly copositive and zero-subhomogeneous.

(c) If $q \in [\mathcal{S}(0, F_c^\infty)]^*$ then $\mathcal{S}(q, F) \neq \emptyset$ (possibly unbounded).

Under the above assumptions,

$$\mathrm{int}[\mathcal{S}(0, F_c^\infty)]^* = [\mathcal{S}(0, F_c^\infty)]^\#.$$

Remark

One can exhibit instances showing that (*a*) in the previous theorem may be false, if either $F \notin \mathbf{T}$, or $q \notin [\mathcal{S}(0, F_c^{\infty}))]^{\#}$ or F_c^{∞} is not a **G**-mapping. Also there is an example showing that the strict copositivity of G or the condition $q \in [\mathcal{S}(0, F_c^{\infty})]^{\#}$ cannot be avoided to obtain the boundedness of the solution set.

Let $d > 0$, $c \in \mathcal{C}$ and $F, \in \mathcal{X}$ such that $A^{\infty}_{F,c}(v) < +\infty \ \forall \ v \in \Delta_d$. The system

$$v \geq 0, \langle d, v\rangle = 1, w \in F^{\infty}_c(v), \langle w, v\rangle \leq 0, w - \langle w, v\rangle d \geq 0, \quad (1)$$

found in the basic Lemma (for $G^k = G$, and $q^k = q$ for all k), plays a fundamental role in characterizing the nonemptiness and boundedness of $\mathcal{S}(q, F + G)$ for all $q \in \mathbb{R}^n$. When F^{∞}_c is c^{∞}-subhomogeneous the inconsistency of (1) is equivalent to the inconsistency of the following system

$$0 \neq v \geq 0, z \in F^{\infty}_c(v), \ \tau \geq 0, z + \tau d \geq 0, \langle z + \tau d, v\rangle = 0. \quad (2)$$

When $F(x) = Mx$ with M being a real matrix the previous system was introduced by Karamardian 1972, giving rise to *regular matrices*.

Asymptotically Regular-type mappings-Robustness property

Definition

Given $d >$ and $c \in \mathcal{C}_0$, we say that F is asymptotically (regular) $\mathbf{R}(d)$-mapping, or shortly $F \in \mathbf{R}(d)$, if $\mathcal{S}(\tau d, F_c^\infty) = \{0\}\ \forall\ \tau \geq 0$.

That is, $F \in \mathbf{R}(d)$ if

$$F_c^\infty \in \mathbf{G}(d) \ \text{ and } \ \mathcal{S}(0, F_c^\infty) = \{0\}.$$

Theorem [FB, 2007]

Let $d > 0$, $c \in \mathcal{C}_0$ satisfying $0 < c^\infty(t) \ \ \forall\, t > 0$, and $F \in \mathcal{X}$ be c-subhomogeneous such that $A^\infty_{F,c}(v) < +\infty \ \forall\, v \in \Delta_d$. Assume in addition that $F^\infty_c \in \mathbf{G}(d)$. The following assertions are equivalent:

- (a) $\mathcal{S}(q, F)$ is nonempty and compact for all $q \in \mathbb{R}^n$;
- (b) $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive satisfying $A^\infty_{G,c}(v) = 0 \ \ \forall\, v \in \Delta_d$;
- (c) $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and uniformly bounded;
- (d) $\mathcal{S}(q, F + G)$ is nonempty and compact for all $q \in \mathbb{R}^n$ and all $G \in \mathcal{X}$ copositive and zero-subhomogeneous;
- (e) $\mathcal{S}(0, F^\infty_c) = \{0\}$.

Sensitivity results

By $\mathcal{H}$ we denote the family of multifunctions with nonempty values, defined on $\mathbb{R}^n_+$, satisfying

$$H(tx) = c^\infty(t)H(x) \;\; \forall\, t > 0,\; \forall\, x \geq 0,$$

with c^∞ such that ($c \in \mathcal{C}_0$)

$$c^\infty(1) = 1,\;\; c^\infty(\xi)c^\infty(\lambda) = c^\infty(\xi\lambda),\; \xi > 0,\;\; \lambda > 0.$$

For $H \in \mathcal{H}$ and $d > 0$, let us consider the *outer norm* [RW]

$$|H|^+_d \doteq \sup\left\{||y|| :\;\; y \in H(x),\;\; x \in \Delta_d\right\},\;\; |H|^+_d < +\infty \;\; \forall\, H \in \mathcal{H}_0.$$

On $\mathcal{H}_0 \doteq \{H \in \mathcal{H} : H \text{ is loc. bound on } \Delta_d, H(0) = 0\}$,

$$|H_1 - H_2|^+_d \doteq \sup\{|y_1 - y_2| : y_1 \in H_1(x),\;\; y_2 \in H_2(x),\;\; x \in \Delta_d\}$$

becomes a metric.

Given $c \in \mathcal{C}_0$, set

$$\mathcal{X}_0 = \Big\{ F \in \mathcal{X} : F_c^{\infty}(0) = \{0\},\ A_{F,c}^{\infty}(v) < +\infty \ \ \forall\, v \in \Delta_d \Big\}.$$

Thus, $F \in \mathcal{X}_0$ implies $F_c^{\infty} \in \mathcal{H}_0$.

Proposition: [FB, 2007]

Let $F^k : \mathbb{R}_+^n \hookrightarrow \mathbb{R}^n$ be any multifunctions with nonempty values, $c \in \mathcal{C}$ and $v \geq 0$. If $\limsup_k{}^{\infty} A_{F^k,c}(v) < +\infty$ then $\limsup_k{}^{\infty,c} F^k$ is locally bounded at v.

Proposition: [FB, 2007]

Let $d > 0$, $c \in \mathcal{C}_0$, $q^0 \in \mathbb{R}^n$, $F^0 \in \mathcal{X}_0$. If $q^0 \in [\mathcal{S}(0, (F^0)^\infty_c)]^\#$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$, all $F \in \mathcal{X}_0$ satisfying

$$||q - q^0|| + |F^\infty_c - (F^0)^\infty_c|_d < \varepsilon$$

one has $q \in [\mathcal{S}(0, F^\infty_c)]^\#$.

Theorem: [FB, 2007]

Let $d > 0$, $c \in \mathcal{C}_0$, $F^0 \in \mathcal{X}_0$. If $q^0 \in [\mathcal{S}(0,(F^0)^\infty_c)]^\#$, then there exists $\varepsilon > 0$ such that for all $q \in \mathbb{R}^n$, all $F \in \mathcal{X}_0$, $\mathbf{T}$ − *mappings*, with $F^\infty_c \in \mathbf{G}(d)$ satisfying

$$||q - q^0|| + |F^\infty_c - (F^0)^\infty_c|_d < \varepsilon,$$

one has $\mathcal{S}(q, F)$ is non-empty and compact.