Regularity of optimal control problems with super linear growth

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Outline

Aim

We investigate the regularity of the value function

$$u(x,t) = \inf \left(\int_t^T L(x(s), s, x'(s))ds + g(x(T)) \right)$$

where the infimum is taken over the $x(\cdot) \in W^{1,1}([t,T], \mathbb{R}^N)$ such that $x(t) = x$

under the key assumption that L has a superlinear growth : $\exists p > 1$, $\delta \in (0,1)$, $M > 0$ with

$$\delta|\xi|^p - M \ \leq \ L(x,s,\xi) \ \leq \ \frac{1}{\delta}|\xi|^p + M$$

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Motivation

Stochastic homogenization of Hamilton-Jacobi equations :

$$(HJ) \qquad u^\epsilon_t(x,t) + H(\frac{x}{\epsilon}, \frac{t}{\epsilon}, Du^\epsilon(x,t)) = 0 \text{ in } \mathbb{R}^N \times (0,T)$$

(Souganidis (1999), Lions-Souganidis (2005), Schwab (2008)).

Recall that our value function u is solution of

$$(HJ) \qquad -u_t(x,t) + H(x,t,Du(x,t)) = 0 \text{ in } \mathbb{R}^N \times (0,T)$$

Known results

Two types of results :

- Propagation of regularity : Lions (1985), Barles (1990), Rampazzo, Sartori (2000)

  If L and g are “sufficiently smooth", then u is Lipschitz continuous with a Lipschitz constant depending on the regularity of L and g.

  Weak point : not suitable for homogenization.

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- Interior regularity/Lipschitz regularity of optimal solutions :
  Clarke, Vinter (1985), Ambrosio, Ascenzi, Buttazzo (1989), Dal Maso, Frankowska (2003), Quincampoix, Zlateva (2006), Frankowska, Marchini (2005), Davini (2007).

  Key assumption : $L = L(x, x')$ has a superlinear growth.

  Consequences :
  - No Lavrentiev phenomenon.
  - Lipschitz continuity of optimal solutions implies Lipschitz continuity of the value function.

  Weak point : Cannot handle the time-dependant case.

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Difficulty for the time-dependent case

If $L = L(x, t, \xi)$, one cannot expect u to be locally Lipschitz continuous because

u Lipschitz continuous $\Rightarrow$ optimal solutions are Lipschitz continuous.

However

Proposition (Lavrentiev (1926))

There are L and initial data (x_0, t_0) for which optimal solutions are not Lipschitz continuous.

Manià's Example (1934) : $L(x, t, \xi) = (t^3 - x)^2 |\xi|^6$.

Assumptions for the main result

We suppose that

- $L = L(x, t, \xi)$ is continuous, convex w.r.t. ξ,
- g is bounded by some $M > 0$,
- (Superlinear growth) there are $p > 1$ and $\delta > 0$ such that

$$\delta|\xi|^p - M \ \leq \ L(x, t, \xi) \ \leq \ \frac{1}{\delta}|\xi|^p + M \qquad \forall (x, t) \in \mathbb{R}^N \times (0, T)$$

Main result

Theorem

There are $\theta > p$ *and, for any* $\tau > 0$, $K_\tau > 0$ *such that*

$$|u(x_0, t_0) - u(x_1, t_1)| \leq K_\tau \left(|x_0 - x_1|^{(\theta - p)/(\theta - 1)} + |t_0 - t_1|^{(\theta - p)/\theta} \right)$$

for any $x_0, x_1 \in \mathbb{R}^N$, *for any* $t_0, t_1 \in [0, T - \tau]$,

where $\theta = \theta(M, \delta, p, T)$ *and* $K_\tau = K(\tau, M, \delta, q, T)$.

Outline

Main steps

Three main steps :

- Step 1 : a strange inequality satisfied by optimal solutions,
- Step 2 : use of the reverse Hölder inequality to get regularity of the optimal solutions,
- Step 3 : this regularity gives the Hölder continuity of u.

Step 1 : a strange inequality

Lemma

There is $A = A(M, \delta, T) \geq 1$ such that, for any optimal solution $\bar{x}$ starting from x_0 at time t_0,

$$\frac{1}{h}\int_{t_0}^{t_0+h} (\alpha(s))^p ds \leq A\left(\frac{1}{h}\int_{t_0}^{t_0+h} \alpha(s) ds\right)^p \qquad \forall h \in [0, T - t_0]$$

where

$$\alpha(s) = |\bar{x}'(s)| + 1$$

Idea of proof : test the optimality of $\bar{x}$ against

$$\tilde{x}(t) = \begin{cases} \frac{\bar{x}(t_0+h)-x_0}{h}(t - t_0) + x_0 & \text{if } t \in [t_0, t_0 + h] \\ \bar{x}(t) & \text{otherwise} \end{cases}$$

Remarks on the inequality

If $\alpha \in L^p(0,1)$ for some $p > 1$, we have from the Hölder inequality :

$$\left(\frac{1}{h}\int_0^h |\alpha|\right)^p \leq \frac{1}{h^p}\left(\int_0^h |\alpha|^p\right) h^{(1-1/q)p} = \frac{1}{h}\int_0^h |\alpha|^p \quad \forall h \in [0,1]$$

So the inequality satisfied by $\alpha(s) = |\bar{x}'(s)| + 1$:

$$\frac{1}{h}\int_{t_0}^{t_0+h} (\alpha(s))^p ds \leq A\left(\frac{1}{h}\int_{t_0}^{t_0+h} \alpha(s) ds\right)^p \qquad \forall h \in [0, T - t_0]$$

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Step 2 : use of the reverse Hölder inequality

Fix $A > 1$ and $p > 1$.

Lemma (reverse Hölder inequality)

There are $\theta = \theta(A,p) > p$ and $C = C(A,p) > 0$ such that, for any $\alpha \in L^p(0,1)$ satisfying

$$\frac{1}{h}\int_0^h |\alpha(s)|^p ds \leq A\left(\frac{1}{h}\int_0^h |\alpha(s)|ds\right)^p \qquad \forall h \in [0,1],$$

one has

$$\int_0^h |\alpha(s)|ds \leq C\|\alpha\|_{L^p}\, h^{1-1/\theta} \qquad \forall h \in [0,1]\ .$$

Remark : This result is a weak form of Gehring's reverse Hölder inequality.

Step 2 (end) : Regularity of optimal solutions

Corollary

There are $\theta > p$ and C such that, for any $x_0 \in \mathbb{R}^N$ and any $t_0 < T$, if $\bar{x}$ is optimal for the initial position x_0 at time t_0, then

$$\int_{t_0}^{t_0+h} |\bar{x}'(s)|ds \leq C(T-t_0)^{1/\theta - 1/p}\, h^{1-1/\theta} \qquad \forall h \in [t_0, T]$$

Remark : this proves that optimal solutions are $1/\theta-$Hölder continuous.

Step 3 : Use of the regularity of optimal solutions

Corollary (Space regularity)

Let $x_0, x_1 \in \mathbb{R}^N$, $t_0 < T$. Then

$$u(x_1, t_0) - u(x_0, t_0) \le \; K_1(T - t_0)^{-(p-1)(\theta-p)/(p(\theta-1))}|x_1 - x_0|^{(\theta-p)/(\theta-1)}$$

where $K_1 = K_1(M, p, T, \delta)$.

Proof : Let $\bar{x}$ be optimal for x_0 and consider

$$\tilde{x}_h(t) = \begin{cases} \frac{\bar{x}(t_0+h)-x_1}{h}(t - t_0) + x_0 & \text{if } t \in [t_0, t_0 + h] \\ \bar{x}(t) & \text{otherwise} \end{cases}$$

which is admissible for x_1.
Use the regularity of $\bar{x}$ and optimize w.r.t. h.

Outline

Reverse Hölder inequality Lemma

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one has

$$\int_0^h |\alpha(s)|ds \leq C\|\alpha\|_{L^p}\, h^{1-1/\theta} \qquad \forall h \in [0, 1]\,.$$

Gehring result

The proof of the reverse inequality Lemma could be achieved through Gehring's result :
Let $p > 1$ and Ω open subset of $\mathbb{R}^N$. Let $\alpha \in L^p(\Omega)$ be such that

$$\fint_Q |\alpha|^p \leq A \left(\fint_Q |\alpha| \right)^p$$

for any cube $Q \subset \Omega$.

Lemma (Gehring (1973))

There are $\theta > p$ and C such that

$$\|\alpha\|_{L^\theta} \leq C \|\alpha\|_{L^p}$$

where $\theta = \theta(A, p)$ and $C = C(A, p)$.

A direct proof of the reverse inequality Lemma

Let

$$\mathcal{E} = \{\alpha \in L^p(0,1)\ ,\ \alpha \geq 0,\ \alpha \text{ satisfies } (*) \text{ and } \|\alpha\|_{L^p} \leq 1\}$$

where

$$(*) \qquad \frac{1}{h}\int_0^h (\alpha(s))^p ds \leq A\left(\frac{1}{h}\int_0^h \alpha(s)ds\right)^p \qquad \forall h \in [0,1]$$

Claim

For any $\tau \in (0,1]$ the problem

$$\xi(\tau) = \max\left\{\int_0^\tau \alpha(s)ds\ ,\ \alpha \in \mathcal{E}\right\}$$

has a unique maximum denoted $\bar{\alpha}_\tau$.

Case of equality in $(*)$: $\alpha \in \mathcal{E}$ satisfies

$$\frac{1}{h}\int_0^h (\alpha(s))^p ds = A\left(\frac{1}{h}\int_0^h \alpha(s)ds\right)^p \qquad \forall h \in [0,1]$$

if and only if

$$\alpha(s) = A^{-1/p}\,\sigma\, s^{\sigma-1} \qquad \forall s \in [0,1]\ ,$$

where σ be a root of the map $s \to s^p - A(1-p+ps)$. Let γ be the smallest positive root. Then $\gamma > 1 - 1/p$.

Claim

$$\xi(\tau) \le C\tau^\gamma \qquad \forall \tau \in [0,1]$$

for some $C = C(p)$.

Remark : This claim proves the Lemma with $\theta = 1/(1-\gamma) > p$.

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Structure of the optimum

Claim

There is $\bar{\tau} > 0$ such that, $\forall \tau \in (0, \bar{\tau})$,

$$\bar{\alpha}_\tau(t) = \begin{cases} a_\tau & \text{on } [0, \tau) \\ b_\tau & \text{on } [\tau, \tau_1) \\ A^{-1/p}\,\gamma\, t^{\gamma-1} & \text{on } [\tau_1, 1] \end{cases}$$

for some $0 < b_\tau \leq a_\tau$ and $\tau < \tau_1 < 1$.

A differential equation for ξ

Claim

ξ is locally Lipschitz continous and satisfies

$$(**) \qquad (-\tau)\xi'(\tau) + \gamma\xi(\tau) = 0 \qquad \text{for a.e. } \tau \in (0, \bar{\tau}) \ .$$

Proof : For $\lambda > 0$, let

$$\alpha_{\lambda\tau}(s) = \bar{\alpha}_\tau(\lambda s) \qquad s \geq 0 \ .$$

Then $\alpha_{\lambda\tau}/\|\alpha_{\lambda\tau}\|_p$ belongs to $\mathcal{E}$. So

$$(***) \qquad \xi\left(\frac{\tau}{\lambda}\right) \geq \frac{\int_0^{\tau/\lambda} \alpha_{\lambda\tau}}{\|\alpha_{\lambda\tau}\|_p} = \frac{\lambda^{1/p-1}\xi(\tau)}{\left(1 + \int_1^\lambda \bar{\alpha}_\tau^p\right)^{1/p}} \ .$$

with an equality for $\lambda = 1$. Deriving $(***)$ at $\lambda = 1$ gives (**).

Generalizations

(work in preparation with P. Cannarsa)

- solutions of HJ equations

$$(HJ) \qquad u_t(x,t) + H(x,t,Du(x,t)) = 0$$

under a superlinear growth condition on H, but no convexity assumption,

- solutions of degenerate parabolic equations of the form

$$(HJ2) \quad u_t(x,t) - \mathrm{Tr}(A(x,t)D^2u(x,t)) + H(x,t,Du(x,t)) = 0$$

under a superquadratic growth condition on H.

Main open question

- What happens if the growth of L is anisotropic ?

$$\delta|\xi|^p - M \leq L(x,t,\xi) \leq \frac{1}{\delta}|\xi|^{p'} + M$$

for some $1 < p < p'$.