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Asymptotic behavior of Multiscaled Gradient Dynamics. Applications to Coupled systems, Games and PDE's.

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SETTING

- H Hilbert space
- $\Phi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ closed convex proper function.
- $\Psi : H \longrightarrow \mathbb{R}^{+} \cup \{+\infty\}$ closed convex proper function, $C = \text{argmin}\Psi = \Psi^{-1}(0) \neq \emptyset$.
- $\beta : \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$ a function of t which tends to $+\infty$ as t goes to $+\infty$.

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

$\Phi + \beta(t)\Psi \uparrow \Phi + \delta_C$ as $t \to +\infty$: (MAG)= "Multiscale Asymptotic Gradient" system.

Claim: Under ad hoc conditions on β, Φ, Ψ,

$$z(t) \longrightarrow z_\infty \in \text{argmin}_C \Phi \text{ as } t \longrightarrow +\infty.$$

Motivation: Dynamic and Algorithmic approach to Optimization and Potential Games:

$$\min \{f(x) + g(y) : \quad Ax - By = 0\}.$$

COUPLED GRADIENT SYSTEMS

- $H = X \times Y$ the cartesian product of two Hilbert spaces, $z = (x, y)$.
- $\Phi(z) = f(x) + g(y)$, $f \in \Gamma_0(X)$, $g \in \Gamma_0(Y)$.
- $\Psi(z) = \frac{1}{2}\|Ax - By\|^2$, A and B linear continuous operators.

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

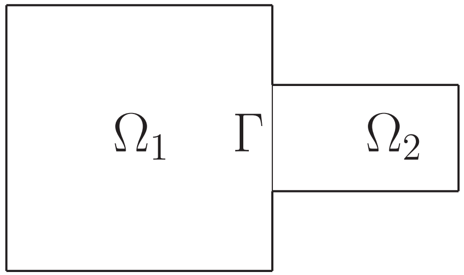
$$\Updownarrow$$

$$\begin{cases} \dot{x}(t) + \partial f(x(t)) + \beta(t)A^t(Ax(t) - By(t)) \ni 0 \\ \dot{y}(t) + \partial g(y(t)) + \beta(t)B^t(By(t) - Ax(t)) \ni 0 \end{cases}$$

Claim: $z(t) = (x(t), y(t)) \rightarrow z_\infty = (x_\infty, y_\infty)$ where (x_∞, y_∞) is a solution of

$$\min \{f(x) + g(y) : \quad Ax - By = 0\}.$$

EXAMPLE 1: DECOMPOSITION OF DOMAINS IN PDE's.



Dirichlet problem on Ω: $h \in L^2(\Omega)$ given, find $z : \Omega \longrightarrow \mathbb{R}$ solution of

$$\begin{cases} -\Delta z = h \ \text{ on } \Omega \\ z = 0 \ \text{ on } \partial\Omega \end{cases}$$

Variational formulation:

$$\min \left\{ \frac{1}{2} \int_{\Omega_1} \nabla z_1|^2 + \frac{1}{2} \int_{\Omega_2} |\nabla z_2|^2 - \int_\Omega hz : \quad z_1 \in X_1, \ z_2 \in X_2, \quad [z] = 0 \quad \text{on } \Gamma \right\}.$$

- $X_i = \{z \in \ H^1(\Omega_i), \quad z = 0$ on $\partial\Omega \cap \partial\Omega_i\}, \quad z = z_i$ on $\Omega_i, \ i = 1, 2.$
- $[z]$ = jump of z through the interface Γ.

$$\min \{f_1(z_1) + f_2(z_2) : z_1 \in X_1, z_2 \in X_2, \quad A_1(z_1) - A_2(z_2) = 0\}.$$

$A_i : H^1(\Omega_i) \to \mathcal{Z} = L^2(\Gamma)$ is the trace operator, $i = 1, 2$.

EXAMPLE 2: POTENTIAL GAMES, BEST RESPONSE DYNAMICS

Static loss functions of players 1 and 2:

$$\begin{cases} F : (\xi, \eta) \in X \times Y \to F(\xi, \eta) = f(\xi) + \beta \Psi(\xi, \eta) \\ G : (\xi, \eta) \in X \times Y \to G(\xi, \eta) = g(\eta) + \mu \Psi(\xi, \eta). \end{cases}$$

Best reply dynamic with cost to change, (players 1 and 2 play alternatively):

$$z_k = (x_k, y_k) \longrightarrow (x_{k+1}, y_k) \longrightarrow z_{k+1} = (x_{k+1}, y_{k+1}) \quad k = 0, 1, \ldots$$

$$\begin{cases} x_{k+1} = \operatorname{argmin}\{f(\xi) + \beta_k \Psi(\xi, y_k) + \frac{\alpha}{2} \| \xi - x_k \|^2_{\mathcal{X}} \colon \xi \in \mathcal{X}\} \\ y_{k+1} = \operatorname{argmin}\{g(\eta) + \beta_k \Psi(x_{k+1}, \eta) + \frac{\nu}{2} \| \eta - y_k \|^2_{\mathcal{Y}} \colon \ \eta \in \mathcal{Y}\} \end{cases}$$

Corresponding continuous dynamical system (MAGS): $z(t) = (x(t), y(t))$

$$\begin{cases} \dot{x}(t) + \partial f(x(t)) + \beta(t) \nabla_x \Psi(x(t), y(t)) \ni 0 \\ \dot{y}(t) + \partial g(y(t)) + \beta(t) \nabla_y \Psi(x(t), y(t)) \ni 0 \end{cases}$$

$\beta(t) \to +\infty$ as $t \to +\infty$ = increasing weight of the cooperative behaviour aspects.

CONTENTS

MULTISCALE FEATURES. SLOW-FAST DYNAMICS 1.

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0$$

is the combination of two dynamics:

- A slow dynamic: (1) $\qquad \dot{z}(t) + \partial\Phi(z(t)) \ni 0.$
- A fast dynamic: (2) $\qquad \dot{z}(t) + \beta(t)\partial\Psi(z(t)) \ni 0.$

Change of time scaling in (2): take $t = \tau(s)$ and set $z(\tau(s)) = w(s)$.

$$(2) \Leftrightarrow \frac{1}{\dot{\tau}(s)\beta(\tau(s))}\dot{w}(s) + \partial\Psi(w(s)) \ni 0.$$

Take $\dot{\tau}(s)\beta(\tau(s)) = 1$, i.e., $\int_0^{\tau(s)} \beta(\xi)d\xi = s$.
Assume $\int_0^{+\infty} \beta(\xi)d\xi = +\infty$, then

$$(2) \Leftrightarrow \dot{w}(s) + \partial\Psi(w(s)) \ni 0.$$

MULTISCALE FEATURES. SLOW-FAST DYNAMICS 2.

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

Change of time scaling: take $t = \tau(s)$ and set $z(\tau(s)) = w(s)$, $\epsilon(s) = \frac{1}{\beta(\tau(s))}$.
Equivalent system with $\epsilon(s) \to 0$ as $s \to +\infty$, $\int_0^{+\infty} \epsilon(s)ds = +\infty$.

$$\dot{w}(s) + \epsilon(s)\partial\Phi(w(s)) + \partial\Psi(w(s)) \ni 0.$$

From $\dot{\tau}(s)\beta(\tau(s)) = 1$, $\dot{\tau}(s) = \frac{1}{\beta(\tau(s))} = \epsilon(s)$, and $\int_0^{+\infty} \epsilon(s)ds = lim_{s\to+\infty}\tau(s) = +\infty$.

Classical situation: $\Phi(w) = \frac{1}{2}\|w\|^2 \Rightarrow$ Asymptotic Tikhonov selection property:

Att.-Cominetti, Att.-Czarnecki, Cabot, Combettes-Hirstoaga, Peypouquet.

ERGODIC CONVERGENCE RESULTS: $\beta(t) \to +\infty$

$$(MAG) \qquad \dot{z}(t) + A(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

- $A : H \to H$ general maximal monotone operator.
- $\Psi : H \to \mathbb{R}^+ \cup \{+\infty\}$ closed convex proper, $C =$ argmin$\Psi = \Psi^{-1}(0) \neq \emptyset$.
  $\Psi^*=$ Fenchel conjugate of Ψ, $\sigma_C=$ support function of C, $N_C=$ normal cone to C.

Theorem 1 [**A.-C.**] Let us assume that,

∞

- $(\mathcal{H}_0)$ $\quad A + N_C$ is a maximal monotone operator and $S := (A + N_C)^{-1}(0) \neq \emptyset$ closed convex set.
- $(\mathcal{H}_1)$ $\quad \forall p \in N_C \quad \int_0^{+\infty} \beta(t) \left[\Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right)\right] dt < +\infty.$

Then,

- $w - lim_{t\to+\infty} \ \frac{1}{t}\int_0^t z(s)ds = z_\infty$ exists with $z_\infty \in S$.
- $\forall a \in S \ \ lim_{t\to+\infty}\|z(t) - a\|^2$ exists.
- $\int_0^{+\infty} \beta(t)\Psi(z(t))dt < +\infty.$

Interpretation of the condition

$$(\mathcal{H}_1) \quad \forall p \in N_C \qquad \int_0^{+\infty} \beta(t) \left[\Psi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \right] dt < +\infty.$$

- Model situation: $\Psi(z) = \frac{1}{2} dist^2(z, C) = \frac{1}{2}\|.\|^2 \nabla \delta_C$.

  $\Psi^*(z) = \frac{1}{2}\|z\|^2 + \sigma_C(z)$ and $\Psi^*(z) - \sigma_C(z) = \frac{1}{2}\|z\|^2$.

$$(\mathcal{H}_1) \Leftrightarrow \int_0^{+\infty} \frac{1}{\beta(t)} dt < +\infty.$$

- $\Psi = 0$. Then $(\mathcal{H}_1)$ is automatically satisfied $(N_C = 0)$ and Theorem 1 $\Leftrightarrow$ Baillon-Brezis ergodic convergence theorem for

$$\dot{z}(t) + A(z(t)) \ni 0$$

  with A maximal monotone operator.

ERGODIC CONVERGENCE RESULTS: $\epsilon(t) \to 0$

Equivalent system with $\epsilon(s) \to 0$ as $s \to +\infty$, $\int_0^{+\infty} \epsilon(s)ds = +\infty$.

$$\dot{w}(s) + \epsilon(s)A(w(s)) + \partial\Psi(w(s)) \ni 0.$$

- $A : H \to H$ general maximal monotone operator.
- $\Psi : H \to \mathbb{R}^+ \cup \{+\infty\}$ closed convex proper, $C = \text{argmin}\Psi = \Psi^{-1}(0) \neq \emptyset$.

Theorem 2 [**A.-C.**] Let us assume that,

- $(\mathcal{H}_0)$ $\quad A + N_C$ is a maximal monotone operator and $S := (A + N_C)^{-1}(0) \neq \emptyset$.
- $(\mathcal{H}_1)$ $\quad \forall p \in N_C \quad \int_0^{+\infty} \left[\Psi^*(\epsilon(s)p) - \sigma_C(\epsilon(s)p)\right] ds < +\infty$.

Then,

$$w - lim_{s\to+\infty} \ \frac{1}{\int_0^s \epsilon(\tau)d\tau} \int_0^s w(\tau)\epsilon(\tau)d\tau = w_\infty \text{ exists with } w_\infty \in S.$$

LINKS WITH PASSTY THEOREM

- $H = X \times Y$ the cartesian product of two Hilbert spaces, $X = Y$, $z = (x, y)$.
- A and B two maximal monotone operators, $M(z) = M(x, y) = (Ax, By)$.
- $\Psi(z) = \frac{1}{2}\|x - y\|^2$ (strong coupling).

$$\dot{w}(s) + \epsilon(s)M(w(s)) + \partial\Psi(w(s)) \ni 0.$$

$$\Updownarrow$$

$$\begin{cases} \dot{x}(s) + \epsilon(s)A(x(s)) + x(s) - y(s) \ni 0 \\ \dot{y}(t) + \epsilon(s)B(y(s)) + y(s) - x(s) \ni 0 \end{cases}$$

Discrete version:

$$\begin{cases} x_{k+1} - x_k + \epsilon(s_k)A(x_{k+1}) + x_k - y_k \ni 0 \\ y_{k+1} - y_k + \epsilon(s_k)B(y_{k+1}) + y_k - x_{k+1} \ni 0 \end{cases}$$

$$y_{k+1} = (I + \epsilon_k B)^{-1}(I + \epsilon_k A)^{-1} y_k$$

Theorem [**Passty, JMMA, 1979**]: Suppose $(\epsilon_k)_{k\in\mathbb{N}} \in l^2(\mathbb{N}) \setminus l^1(\mathbb{N})$, then

$$z_n = \frac{1}{\sum_1^n \epsilon_k} \sum_1^n \epsilon_k y_k \to z_\infty \text{ weakly in } X \text{ with } Az_\infty + Bz_\infty \ni 0.$$

FROM ERGODIC CONVERGENCE TO CONVERGENCE: $\beta(t) \to +\infty$

Take $A = \partial\Phi$ a subdifferential operator, and use energy estimates.

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

Theorem 3 [**A.-C.**] Let us assume

- $(\mathcal{H}_0)$, $(\mathcal{H}_1)$.
- $(\mathcal{H}_2)$ $\quad \beta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a smooth (C^1) increasing function and there exists some positive constant $k > 0$ such that for t large enough:

$$\dot{\beta}(t) \leq k\beta(t).$$

Then,

- $w - lim_{t \to +\infty}\, z(t) = z_\infty$ exists with $z_\infty \in S$.
- $lim_{t \to +\infty}\, \Psi(z(t)) = 0$.
- $lim_{t \to +\infty}\, \Phi(z(t)) = inf_C \Phi$.

STRONG CONVERGENCE RESULTS

$$(MAG) \qquad \dot{z}(t) + A(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

- $A : H \longrightarrow H$ maximal monotone operator which is strongly monotone, i.e.,

$$\exists \alpha > 0 \text{ such that } \langle Au - Av, u - v\rangle \geq \alpha \|u - v\|^2 \ \forall u, v \in H.$$

- $\Psi : H \longrightarrow \mathbb{R}^+ \cup \{+\infty\}$ closed convex proper, $C = \text{argmin}\Psi = \Psi^{-1}(0) \neq \emptyset$.

Theorem 4 [**A.-C.**] Let us assume that A is a strongly monotone operator and

- $(\mathcal{H}_0) \quad A + N_C$ is a maximal monotone operator.
- $\beta(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$.

Then,

- $S = (A + N_C)^{-1}0$ is reduced to a single element $\overline{z}$.
- $s - lim_{t\rightarrow+\infty} \ z(t) = \overline{z}$.

CONVERGENCE RESULTS: THE FINITE DIMENSIONAL CASE

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

Equivalent system with $\epsilon(s) \to 0$ as $s \to +\infty$, $\int_0^{+\infty} \epsilon(s)ds = +\infty$:

$$\dot{w}(s) + \epsilon(s)\partial\left[\Phi + \frac{1}{\epsilon(s)}\Psi\right](w(s)) \ni 0.$$

- $\Phi + \frac{1}{\epsilon(s)}\Psi$ Mosco epi-converges to $\Phi + \delta_C$ as $s \to +\infty$.

- $\Phi + \frac{1}{\epsilon(s)}\Psi$ converges uniformly to $\Phi + \delta_C$ on $S = \text{argmin}\,(\Phi + \delta_C)$.

From Baillon and Cominetti, J. Funct. Analysis (2001)

$$dist(w(s), S) \text{ tends to } 0 \text{ as } s \to +\infty.$$

Combining with the Fejer monotonicity property (valid under $(\mathcal{H}_1)$) + Opial's lemma $\Rightarrow$:

$$(MAG) \qquad \dot{z}(t) + \partial\Phi(z(t)) + \beta(t)\partial\Psi(z(t)) \ni 0.$$

Theorem 5 [**A.-C.**] Let us assume that

- H is a finite dimensional space and $S = \text{argmin}_C \Phi$ is a bounded set.
- $\beta(t) \longrightarrow +\infty$ as $t \longrightarrow +\infty$.
- $(\mathcal{H}_1) \quad \forall p \in N_C \quad \int_0^{+\infty} \beta(t) \left[\Psi^*(\frac{p}{\beta(t)}) - \sigma_C(\frac{p}{\beta(t)}) \right] dt < +\infty.$

Then,

- $lim_{t \to +\infty} \, z(t) = z_\infty$ exists with $z_\infty \in S$.
- $lim_{t \to +\infty} \, \Psi(z(t)) = 0$.
- $lim_{t \to +\infty} \, \Phi(z(t)) = inf_C \Phi$.

CONVERGENCE RESULTS: $\epsilon(t) \to 0$

Equivalent system with $\epsilon(s) \to 0$ as $s \to +\infty$, $\int_0^{+\infty} \epsilon(s)ds = +\infty$:

$$\dot{w}(s) + \epsilon(s)\partial\Phi(w(s)) + \partial\Psi(w(s)) \ni 0.$$

- $\Phi : H \to \mathbb{R} \cup \{+\infty\}$ closed convex proper function.
- $\Psi : H \to \mathbb{R}^+ \cup \{+\infty\}$ closed convex proper, $C =$ argmin$\Psi = \Psi^{-1}(0) \neq \emptyset$.

Theorem 6 [**A.-C.**] Let us assume that,

- $(\mathcal{H}_0)$ $\partial\Phi + N_C$ is a maximal monotone operator and $S := (\partial\Phi + N_C)^{-1}(0) \neq \emptyset$.
- $(\mathcal{H}_1)$ $\forall p \in N_C \quad \int_0^{+\infty} [\Psi^*(\epsilon(s)p) - \sigma_C(\epsilon(s)p)]\, ds < +\infty$.
- $(\mathcal{H}_2)$ There exists some $k > 0$ such that for s large enough

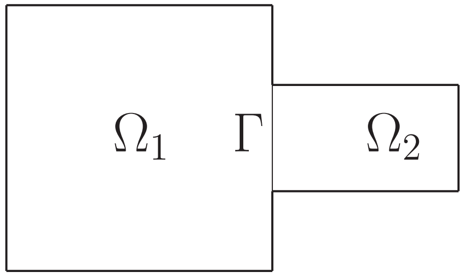
$$-\frac{\dot{\epsilon}(s)}{\epsilon^2(s)} \leq k.$$

Then,

$$weak - lim_{t\to+\infty}\, w(t) = w_\infty \quad \text{exists with} \quad w_\infty \in S.$$

$$lim_{t\to+\infty}\, \Psi(w(t)) = 0, \quad lim_{t\to+\infty}\, \Phi(w(t)) = inf_C\Phi.$$

DECOMPOSITION OF DOMAINS IN PDE's.



Dirichlet problem on Ω: $h \in L^2(\Omega)$ given, find $u : \Omega \longrightarrow \mathbb{R}$ solution of

$$\begin{cases} -\Delta u = h \ \text{ on } \Omega \\ u = 0 \ \text{ on } \partial\Omega \end{cases}$$

Variational formulation:

$$\min \left\{ \frac{1}{2} \int_{\Omega_1} \nabla v_1|^2 + \frac{1}{2} \int_{\Omega_2} |\nabla v_2|^2 - \int_{\Omega} hv : \quad v_1 \in X_1,\ v_2 \in X_2, \quad [v] = 0 \quad \text{on } \Gamma \right\}.$$

- $X_i = \{v \in \ H^1(\Omega_i), \quad v = 0$ on $\partial\Omega \cap \partial\Omega_i\}, \quad v = v_i$ on $\Omega_i, \ i = 1, 2.$
- $[v] =$ jump of v through the interface Γ.

$$\min \left\{ f_1(v_1) + f_2(v_2) : v_1 \in X_1, v_2 \in X_2, \quad A_1(v_1) - A_2(v_2) = 0 \right\}.$$

$A_i : H^1(\Omega_i) \to \mathcal{Z} = L^2(\Gamma)$ is the trace operator, $i = 1, 2.$

Continuous dynamical system:

$$\begin{cases} -\Delta \dfrac{\partial u_1}{\partial t} - \Delta u_1 = h_1 \text{ on } \Omega_1 \\ -\Delta \dfrac{\partial u_2}{\partial t} - \Delta u_2 = h_2 \text{ on } \Omega_2 \\ \dfrac{\partial \dot{u}_1(t)}{\partial \nu_1} + \dfrac{\partial u_1}{\partial \nu_1}(t) - \beta(t)\left[u(t)\right] = 0 \text{ on } \Gamma \\ \dfrac{\partial \dot{u}_2(t)}{\partial \nu_2} + \dfrac{\partial u_2}{\partial \nu_2}(t) + \beta(t)\left[u(t)\right] = 0 \text{ on } \Gamma \end{cases}$$

Discrete version: Alternating Algorithm with Dirichlet-Neumann transmission conditions:

$$(u_{1,k}, u_{2,k}) \longrightarrow (u_{1,k+1}, u_{2,k}) \longrightarrow (u_{1,k+1}, u_{2,k+1}) \text{ with } \beta_k \rightarrow +\infty.$$

$$\begin{cases} -(1+\alpha)\Delta u_{1,k+1} = h_1 - \alpha \Delta u_{1,k} \text{ on } \Omega_1 \\ (1+\alpha)\dfrac{\partial u_{1,k+1}}{\partial \nu_1} + \beta_k u_{1,k+1} = \beta_k u_{2,k} + \alpha \dfrac{\partial u_{1,k}}{\partial \nu_1} \text{ on } \Gamma \\ u_{1,k+1} = 0 \text{ on } \partial\Omega_1 \cap \partial\Omega \end{cases}$$

$$\begin{cases} -(1+\alpha)\Delta u_{2,k+1} = h_2 - \alpha \Delta u_{2,k} \text{ on } \Omega_2 \\ (1+\alpha)\dfrac{\partial u_{2,k+1}}{\partial \nu_2} + \beta_k u_{2,k+1} = \beta_k u_{1,k+1} + \alpha \dfrac{\partial u_{2,k}}{\partial \nu_2} \text{ on } \Gamma \\ u_{2,k+1} = 0 \text{ on } \partial\Omega_2 \cap \partial\Omega \end{cases}$$

POTENTIAL GAMES AND BEST RESPONSE DYNAMICS

Static loss functions of players 1 and 2:

$$\begin{cases} F : (\xi, \eta) \in X \times Y \to F(\xi, \eta) = f(\xi) + \beta\Psi(\xi, \eta) \\ G : (\xi, \eta) \in X \times Y \to G(\xi, \eta) = g(\eta) + \mu\Psi(\xi, \eta). \end{cases}$$

Best reply dynamic with cost to change, (players 1 and 2 play alternatively):

$$(x_k, y_k) \longrightarrow (x_{k+1}, y_k) \longrightarrow (x_{k+1}, y_{k+1}) \quad k = 0, 1, \ldots$$

$$\begin{cases} x_{k+1} = \text{argmin}\{f(\xi) + \beta_k\Psi(\xi, y_k) + \frac{\alpha}{2} \parallel \xi - x_k \parallel^2_{\mathcal{X}} : \xi \in \mathcal{X}\} \\ y_{k+1} = \text{argmin}\{g(\eta) + \beta_k\Psi(x_{k+1}, \eta) + \frac{\nu}{2} \parallel \eta - y_k \parallel^2_{\mathcal{Y}} : \ \eta \in \mathcal{Y}\} \end{cases}$$

Corresponding continuous dynamical system (MAGS):

$$\begin{cases} \dot{x}(t) + \partial f(x(t)) + \beta(t)\nabla_x\Psi(x(t), y(t)) \ni 0 \\ \dot{y}(t) + \partial g(y(t)) + \beta(t)\nabla_y\Psi(x(t), y(t)) \ni 0 \end{cases}$$

$\beta(t) \to +\infty$ as $t \to +\infty$ = increasing weight of the collective behaviour aspects.

PERSPECTIVES

- $\dot{\beta}(t) \leq k\beta(t)$ optimal? Examples, counterexamples.
- Is (MAG) asymptotically almost-equivalent to an autonomous gradient-like system (gradient-projection...)?
- Can one drop the qualification assumption? (variational sum of monotone operators...)
- From Penalization to Lagrangian and augmented Lagrangian methods.
- From first to second order time derivatives differential systems: control of oscillations, synchronization.
- From continuous to discrete dynamics. Passty theorem with general (weak) coupling .
- From convex to non-convex analysis. Kurdyka-Lojasiewicz inequality for gradient systems.
- Parallel computing via forward-backward algorithm, Att-Bruceno-Combettes.
- Numerical developments for domain decomposition problems.
- Potential games, coordination games, learning cooperative behaviours.

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